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Students' mathematics-related belief systems and their strategies for solving non-routine mathematical problems

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Abstract

This study explored students' mathematics-related beliefs and the relationship between the beliefs and their strategies for solving non-routine mathematical problems. The study was guided by Daskalogianni and Simpson's 2001 belief systems categories and strategies for non-routine mathematical problems. The participants were 625 grade 11 students from five high schools in Tshwane North District, Gauteng province of South Africa. Data were collected using a mathematics beliefs questionnaire, a mathematics problem-solving test and interview. Quantitative and qualitative research techniques were used for data analysis. It was found that the students held all the three belief systems (utilitarian, systematic and exploratory) at different degrees of intensity and the belief systems and strategies for problem-solving had a weak positive linear relationship, and there were no statistically significant differences among mean scores of the students holding systematic, exploratory and utilitarian beliefs. They apply unsystematic guess, check and revise; systematic guess, check and revise; systematic listing; looking for patterns; consider a simple case; modelling; logical reasoning; no logical reasoning; trial-and-error and use a formula in solving non-routine mathematical problems. Furthermore, it was found that the systematic belief system could explain the students' behaviour in problem-solving more than the exploratory and utilitarian belief systems.

Keywords

Mathematics-related belief system; non-routine problems; problem-solving strategies

Introduction

Mathematics-related beliefs can be defined as an individual's subjective knowledge of mathematics education, the self as a mathematician and the mathematics class context. An individual's beliefs do not exist in isolation from each other but in clusters, which altogether constitute their belief system. A belief system can be composed of both positive or healthy beliefs (beliefs that promote effective problem-solving) and negative or unhealthy beliefs (beliefs that hinder effective problem-solving). Belief systems are considered one of the significant factors that affect mathematics problem-solving (Ozturk, & Guven, 2016; Yunmin, 2014, Schoenfeld, 2016). They, to a greater extent, influence students'



determination of strategies or techniques to apply or not to apply to problem-solving, self-evaluation of their mathematics problem-solving capabilities, evaluation of time to invest on problem-solving and the methods to employ on monitoring and controlling the problem-solving process (Schoenfeld, 2016). Thus, becoming better problem solvers entails amending students' beliefs about mathematics (Gijsbers et al., 2019; Maiorca, 2016; Prendergast et al., 2018; Frank, 1988). Students' negative beliefs seem to have long-lasting adverse effects on their academic career by affecting proper adaptation to each level of education, stretching from primary, through secondary to tertiary education.

Though several factors have been identified as possible causes for students' poor performance on non-routine questions (Department of Basic Education [DBE], 2019; Wessels, 2009; Reddy et al., 2006; Maree et al., 2006), the effect of students' mathematics-related beliefs to mathematics problem-solving is somehow ignored in the South African mathematics classrooms. This study focused on the relationship between belief systems and approaches to problem-solving to raise an awareness of the importance of belief systems on mathematics learning and achievement. The assessment of students' mathematics-related beliefs can assist educators on planning and structuring the classroom environment in ways that promote students' development of healthy beliefs (Spangler, 1992). Knowledge of students' existing belief systems can facilitate educators to set tasks that can challenge and enrich their belief systems, and thereby improve performance in mathematics. In this vein, Gawlick (2018) argues that students' belief systems could be used to predict their possible mathematical problem-solving behaviour and achievement.

Due to the possible interaction between belief systems and mathematics problem-solving, there had been a focus among mathematics education scholars on unravelling the relationship between students' beliefs and their achievement in mathematics (see, for example, Bonne, 2016; Beswick, 2011; Hassi & Laursen, 2009), and between beliefs and approaches to mathematics problem-solving (see Ozturk & Guven, 2016; Callejo & Vila, 2009; De Corte & Op't Eynde, 2002). However, some of these studies depended heavily on use of belief scales, while some depended heavily on analysing students' written texts, neglecting students' espoused beliefs that could be tapped by use of interviews. In addition, though there is a shift from studying the relation between single beliefs and problem-solving to studying relations between belief systems and problem-solving, there is an inadequate number of studies done on this field. The findings reached were contradictory and, thus, inconclusive, probably due to use of different belief systems frameworks. In an attempt to advance the findings of studies on this field to a conclusive state, we employ comprehensive research methodologies; use of closed and open form questionnaires, retrospective questionnaires, clinical interviews and mathematics problem-solving tests, to unravel the possible relationship between belief systems and approaches to non-routine mathematical problem-solving. We adopted Daskalogianni and Simpson's (2001) belief systems framework because of its comprehensive inclusion of beliefs pertaining to the nature of mathematics and mathematics problem-solving. We categorised students' belief systems into systematic, utilitarian and exploratory.

The systematic belief system includes beliefs on mathematics as a methodical and logical subject, mathematics exercises following some series of steps, mathematics problems having exact answers, and students depending on notes and the teacher to learn. The utilitarian belief system includes beliefs on mathematics as applicable to other subjects and everyday life, exercises having exact answers, problems solved by use of known algorithms, and students depending on the teacher to learn. The exploratory belief system includes beliefs on mathematics as involving problem-solving and linking things, exercises focusing on exploring challenging situations, and students depending on their own abilities to learn.

The researchers draw on De Corte and Op't Eynde's (2002) proposition that studying belief systems may lead "to a more comprehensive understanding of how beliefs influence mathematics learning and problem-solving" (p. 97). Accordingly, this study is a correlational study of 11th graders' mathematics-related belief systems (MRBS) and their strategies for non-routine mathematical problem-solving (NMPS). It addresses the following research questions: (i) What are the grade 11 students' predominant mathematics-related belief systems? (ii) What are the grade 11 students' strategies for solving non-

routine mathematical problems? (iii) Is there a significant difference in non-routine problem-solving mean scores among students holding different belief systems? (iv) Is there a relationship between the strategies for NMPS and MRBS?

Theoretical background

Mathematics-related belief systems

It is difficult to define beliefs because they are abstract, subjective and products of cultural relativism (see, for example, Callejo & Vila, 2009; Goldin et al., 2009; Leder, 2019). Literature on beliefs shows that mathematics education researchers disagree on both the definition of beliefs and their categories (Benbow, 1995). Lindenskov and Hetmar (2009), for instance, argue that researchers disagree on whether belief should be regarded as a phenomenon or as situated process and action. As such, they end up formulating their definitions with glaring contradictions (Furinghetti & Pehkonen, 2002; Leder, 2019; Pehkonen & Torner, 1999).

An example is that of Furinghetti and Pehkonen (2002), who view beliefs as static, while researchers such as Callejo and Vila (2009), Lazim et al. (2004) and Schoenfeld (2016) posit that beliefs are dynamic. The researchers define mathematics-related beliefs as an individual's subjective knowledge of the world. Such knowledge is constructed from an experience that guides and shapes one's behaviour in mathematics problem-solving. Pointedly, students' beliefs depend largely on their social experiences. This makes their beliefs dynamic and fluid, considering that they are influenced by everyday experiences as they interact with other students, teachers, parents, etc.

Some researchers define a "belief system" by analysing its constituent parts (e.g., Op't Eynde et al., 2006; Schoenfeld, 1985), while others analyse how beliefs in a belief system are structured (e.g., Benbow, 1995; Callejo & Vila, 2009). Drawing on the definitions by Op't Eynde et al. (2006) and Schoenfeld (1985), MRBS is defined here as constituting subjective conceptions students hold about mathematics, students of mathematics and the classroom environment that affect the way they approach problem-solving.

Moreover, there are different categories of mathematics-related belief systems (see, for example, Daskalogianni & Simpson, 2001; Jin, Feng et al., 2010; Op't Eynde et al., 2006). For instance, Brunning et al., (1999) categorised beliefs into a dualist view (knowledge is viewed as either right or wrong), a multiplist view (knowledge is sometimes right and sometimes wrong) and a relativist view (truth of knowledge depends on its context). Bishop (1996) categorised beliefs into absolutist (knowledge is certain, absolute, universally true and static) and fallibilists (knowledge is uncertain and unstatic). However, there is lack of uniformity in the way the mathematics-related belief systems have been categorised. Therefore, this study adopted Daskalogianni and Simpson's (2001) belief systems framework because of its comprehensive inclusion of pertinent mathematics-related beliefs.

Strategies for non-routine mathematics problem-solving

Several researchers have suggested similar approach heuristics to problem-solving (see, for example, Burton, 1984; Cherry, 2011; Polya, 1985). Polya (1985), for instance, suggests a dynamic and cyclic four-stage approach to problem-solving, namely understanding the problem, devising a plan, carrying out the plan and looking back. Some researchers discovered that, with limited competence and effectiveness, students could apply a variety of problem-solving strategies such as Unsystematic Guess, Check and Revise; Systematic Guess, Check and Revise; Systematic Listing; Looking for Patterns, to mention a few (Hang & Wang, 2017; Mabilangan et al., 2011; Mogari & Lupahla, 2013; Szabo et al., 2020). With the benefit of these scholars, the researchers define problem-solving as a process wherein previously acquired knowledge is applied to new and unfamiliar problems to obtain an appropriate

solution. What the researchers consider important in problem-solving are the methods, procedures, strategies and heuristics that students use. Hence, this study focused on strategies for mathematical problem-solving.

Belief systems and mathematics problem-solving

There has been much interest in the effects of beliefs on performance in mathematics (Hassi & Laursen, 2009; Goldin et al., 2009; Hardin, 2003). Hardin (2003) warns that the mathematics-related belief systems a student holds might hinder effective retrieval of relevant information from memory when solving a problem. At a given situation, beliefs may influence decisions on the choice of strategies applicable to problem solving and level of engagement on problem-solving. Mason (2003) has shown that beliefs can act as predictors of achievement in mathematics. Based on knowledge of a student's predominant belief system, we can possibly predict if the student is likely to perform well or bad in mathematics. Studies by Bonne (2016), Doman and Adams (2004), Hassi and Laursen (2009) and Mason (2003) have found beliefs and achievement in mathematics to be related. Students with predominant negative beliefs were associated with poor achievement, while positive beliefs were associated with good performance. Callejo and Vila (2009) identified two possible factors that explain students' problem-solving behaviour, namely dualistic belief systems that emerge from schooling, and motivation to solve a problem that emerges from beliefs about how difficult a problem is. Furthermore, they discovered the existence of a complex relationship between approaches to non-routine mathematics problem-solving and mathematics-related belief systems. As such, they could not conclude whether belief systems affect the approach to problem-solving or vice versa.

Unfortunately, these studies fall short of providing plausible insight into the extent of research on the relationship between belief systems and strategies for problem-solving. Further complications are pointed out by Di Martino (2004) that a single belief may belong to different belief systems in different students and it is this single belief that may bring about different problem-solving behaviours in different students. Callejo and Vila (2009) narrowed their study on two belief categories: The nature of a mathematics problem and the nature of the problem-solving activity that we consider inadequate to have a comprehensive understanding of how mathematics-related belief systems are related to approaches to problem-solving. Based on their findings, Callejo and Vila (2009) posit that to better understand a student's behaviour in mathematical problem-solving it is advisable to study belief systems rather than specific beliefs. It is for these reasons that the interest and focus of this study were belief systems. Daskalogianni and Simpson (2001) analysed students' espoused beliefs solely for the purpose of categorising them. No further study in literature was documented that further advanced this work. Thus, we adopted their belief systems framework and used quantitative and qualitative research methodologies that could be deemed adequate to reveal and analyse the possible relationship between belief systems and approaches to problem-solving.

Method

The study followed a mixed-method. It involved a convenience sample of 625 grade 11 students drawn from five high schools in Tshwane North District, Gauteng province of South Africa. The students completed a mathematics beliefs survey first, following which they sat a mathematics problem-solving test. This approach ensured that the students' beliefs about mathematics were not influenced by the cognitive demands posed by the test questions. Three students out of the 625, whom the researchers thought represented the students of different belief systems, were selected to participate in the interviews and answer open-ended questionnaires.

As such, two questionnaires were used. The first mathematics beliefs questionnaire (MBQ) was closed-form, consisting of 60 items. Guided by Daskalogianni and Simpson's (2001) belief systems

framework, we selected the items from the MBQs of Muis (2004), Op't Eynde et al. (2006) and Physick (2010) (see extract of the MBQ in Appendix A). It was grouped into five factors with Cronbach reliability coefficients (α) of 0,867; 0,794; 0,832; 0,758 and 0,680 respectively using Principal component factor analysis. All except factor 5 had high acceptable reliability coefficients. Factor 5's lower $\alpha = 0,680$ could be attributed to a small number of items in F5 which was 6, while other factors had the number of items ranging from 11 to 16.

The second MBQ was open-ended, consisting of seven questions on definitions of terms (e.g., mathematics problem, problem solving), types of questions encountered in classroom exercises, and students' likes or dislikes about mathematics. The respondent is expected to describe, explain, and justify their responses. The responses to these questions could reveal students' beliefs related to mathematics. The mathematics problem-solving test (MPT) consisted of six real everyday life problems which could be solved by the application of basic mathematics (see Appendix B). As such, the problems were within the reach of grade 11 students. Solutions to the MPT could be used to identify students' problem-solving strategies and infer the underlying beliefs in problem-solving. Five high school mathematics teachers, two of whom were heads of mathematics departments, validated the content of the MPT. The Spearman-Brown reliability coefficient was 0,38. This reliability coefficient was lower than the acceptable level of at least 0,70 (Maree et al., 2006). The researchers attributed the low reliability coefficient to poor performance on the test by most students.

Two interview schedules were used. The first schedule consisted of five basic questions that followed up some students' responses to the MBQ. It asked a student to state reasons why they selected a specific rating for each belief item. The approach followed Di Martino's (2004, p. 277) argument that "the reasons provided by the respondents allow the researcher to highlight other beliefs linked to the declared belief and the psychological centrality of the declared beliefs, thus giving information about the belief system containing it". Schedule 2 (IS2) consisted of eight open questions aimed at gaining a deeper understanding of the students' declared beliefs. Examples of questions asked in IS2 were: What is the best way you think you can learn mathematics? Why do you think maths is or is not important to you? The data obtained from the semi-structured interviews and the open-ended questionnaire was cross-validated to identify, analyse and interpret existing converging and/or diverging responses. As such, "consistency of responses" was used to determine the reliability of the interview schedules.

Analysis

The MBQ items were classified into three belief systems: exploratory, systematic and utilitarian (see Daskalogianni & Simpson, 2001). The responses to MBQ were analysed by calculating each student's mean response to each belief category. The negative items in the questionnaire were coded in the reverse order of the positive items (see Appendix A). A student was considered to hold a belief if their mean score on a given belief category was more than 3 out of 5, and a neutral belief if their mean score was 3 out of 5. The belief category with the highest mean score was the student's predominant belief system that largely influenced their behaviour in problem-solving (Jin et al., 2010; Schoenfeld, 1985).

The students' responses to the MPT were analysed to identify the strategies used to solve the problems. The identified strategies were coded using the coding scheme shown in Table 1. Some students' beliefs were inferred from their written responses to MPT, as these could have influenced their behaviour on solving the problems.

Table 1. Codes Used for Problem-Solving Strategies

Strategy	Code	Description
Systematic listing	SL	The student searches for the solution by using an equally spaced list of at least three values.
Modelling	MD	The student models the problem situation using algebraic equations, inequalities, diagrams, tables etc.
Trial-and-Error	TE	The student searches for the solution by using at least 2 randomly selected values which are not necessarily equally spaced.
Guess, Check and Revise [systematic(sys)/ unsystematic(unsys)]	GCR	Sys:- The student makes a reasonable guess and then checks and revises it if necessary. Unsys: - The student makes a guess and fails to check or revise it. The student presents an answer only, without showing any working.
Use a formula	F	The student uses a formula.
Logical reasoning	LG	The student reasons logically to reach conclusions and justify their statements.
No logical reasoning	NLG	The student presents illogical statements. Student presents an answer that does not answer the problem.
Look for patterns	LP	The student identifies a pattern and deduces a general statement that can be used to solve the problem.
Consider a simple case	SC	The student rewrites/reproduces/rewords the problem statements given. The student uses smaller numbers or breaks down the problem situation into simpler problems.

Adapted from Elia et al., (2009) and Mabilangan, et al (2011).

The problem-solving strategies used by the students were weighted on a 5-point scale (see Appendix B). As such, the researchers used Appendix B as a marking rubric in this study. To add more clarity on the data obtained from MBQ and inferences made on the MPT, verbatim transcripts of interviews and responses to the open-ended questionnaire (OQ) were used. Pearson's correlation coefficient was computed to analyse the relationship between the two constructs. The students' belief systems were examined with their strategies for problem-solving. The contents of the mathematics problem-solving tests were validated by five high school mathematics teachers. The Spearman-Brown reliability coefficient was calculated using SPSS to measure the reliability of the mathematics problem-solving test. Cronbach's reliability coefficient was computed using Principal component factor analysis to determine the reliability of the belief scales. In addition, the reliability of the belief scales were assessed through methodological triangulation and data triangulation. The researchers used questionnaires, mathematics test and interviews to confirm findings and assess the quality of the data obtained. Similarities or consistency on responses from these different measuring tools were used to assess the reliability of the interview schedules. To assess the reliability of the findings, the researchers also used theoretical triangulation by checking the points of agreement or disagreement of their findings and interpretations with those of other researchers on the same field of study.

Findings

Grade 11 students' predominant mathematics-related belief systems (MRBSs)

Based on the analysis of each student's MRBS mean score, 68.2% of the students held the belief items examined in this study. The proportion of students who were neutral was 28.5% (see Table 2).

Table 2. *Beliefs Mean Score Frequencies (N = 625)*

Mean score (nearest unit)	Frequency	Percent
2	21	3.4 %
3	178	28.5 %
4	415	66.4 %
5	11	1.8 %
Total	625	100.0 %

Most students (93.3%) held at least two belief systems (see Table 3). This finding accords with that of Daskalogianni and Simpson (2001), who discovered that students held all the three belief systems (utilitarian, systematic and exploratory) at different degrees of intensity.

Table 3. *Number of Belief Systems Held by a Student (N = 625)*

	Frequency	Percent
0	9	1.4%
1	33	5.3%
2	92	14.7%
3	491	78.6%
Total	625	100.0%

Out of the 625 students, 63.4% were exploratory believers, 26.9% were utilitarian believers, 8% were systematic believers, 1.6% were neutral believers and 0.2% were either systematic or exploratory believers. Most students (90.3%) were exploratory and utilitarian believers. To have an in-depth study of the constituents of each belief system, the researchers analysed the extracted commonalities (see Table 4) and selected for interviews three students who, from the 625 participants, had different predominant belief systems.

Systematic believers' beliefs

Examples of beliefs held strongly by the systematic believers were: "Solving a mathematics problem is demanding and requires thinking from every student (23.8%)" and "Those who are good at mathematics

can solve any problem in a few minutes (21.2%).” The proportions of variation in these variables explained by the other 59 variables in the questionnaire were 23.8% and 21.2%, respectively. The examples of variables that the systematic believers considered less important were “I am not the type to do well in mathematics (47.9%)” and “Mathematics has always been my worst subject (52.3%).” The 47.9% and 52.3% variance could be accounted for by the other 59 variables (see Table 4).

Table 4. *Extract of the “Systematic Beliefs” Questionnaire Items*

	Initial	Extraction
Solving a maths problem is demanding and requires thinking, also from smart students	1.000	.238
I am not good at maths	1.000	.467
Maths has always been my worst subject	1.000	.523
I am not the type to do well in maths	1.000	.479
Those who are good in maths can solve any problem in a few minutes	1.000	.212

Extraction Method: Principal Component Analysis.

The researchers interviewed student HA37 (systematic believer) to gain more insight into systematic believers’ mathematics-related beliefs. HA37 described a mathematical problem as “a problem that needs to be solved by applying a suitable mathematical method and laws”. This description of a mathematical problem reveals a belief that a mathematical problem can be resolved by the mere application of some known mathematical laws, rules, techniques and methods. HA37 prefers solving problems that can be solved using a formula. In this regard, they said: “... with a formula it is like I have the recipe to solve the problem. All I need are the ingredients to apply to solve the problem successfully.” The student believes that mathematics can be learnt by memorising facts and procedures and practising their application in problem-solving to avoid forgetting them. The student considers mathematics as a prerequisite to study for their future career and can widen the range of career choices in life. The following dialogue reveals these beliefs:

- Interviewer: What is the best way you think you can learn mathematics?
- HA37: By practising problem-solving a lot, memorising the laws and facts about mathematics, and using them, where they apply, to solve problems.
- Interviewer: Could you tell me why you “strongly agree” with the item “To me mathematics is an important subject”.
- HA37: To me, mathematics is very important not only because I need it in my future career, but it is a door opener to other career paths.

Exploratory believers’ beliefs

Some variables, and their commonalities, that exploratory believers attached great importance to were “I always prepare myself carefully for exams (24.2%)” and “Mathematics is continuously evolving. New things are still being discovered (18.8%).” Examples of the variables that the exploratory believers considered relatively less important were: “I like doing mathematics (58.7%)” and “I try to play around with ideas of my own and relate them to what I am learning in this course (51.1%).”

Interviews with student HD8 (exploratory believer) revealed that the student views mathematics as a “subject that deals with solving problems”. The student relies on their ability and the teacher’s assistance in learning mathematics. HD8 believes that making mistakes is part of learning mathematics.

They also believe that mathematics could be best learnt through practice on problem-solving and making sense of what one learns. These beliefs are evident in the following dialogue:

Interviewer: Could you tell me why you strongly disagree with the item “It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem”.

HD8: I strongly disagree because I always try to solve new problems on my own so that I can seek help from the teacher on the problems I encounter.

Interviewer: Could you tell me why you strongly agree with the item “Making mistakes is part of learning mathematics”.

HD8: I strongly agree because if I make a mistake on solving a mathematics problem, the teacher will rectify my mistakes and I will not repeat the same mistakes again.

Utilitarian believers' beliefs

The utilitarian believers attached great importance to the following variables: “To me, mathematics is an important subject (21.2%)” and “Group work facilitates the learning of mathematics (21.4%).” Examples of the beliefs that the utilitarian believers considered relatively less important were: “Mathematics is my favourite subject (54.7%)” and “I ask myself questions to make sure I understand the material I have been studying in this class (43.6%)”.

Interviews with student HA27 (utilitarian believer) revealed the following beliefs: There are always numbers in the formulation of mathematical problems. Solving a mathematical problem is a process that involves the analysis of the problem situation and getting a solution to the problem. There is more than one way of solving a mathematical problem. There is some degree of uncertainty in a solution to a problem. As such, there is always room to improve one's solution to a problem. These beliefs are evident in the following dialogue:

Interviewer: What do you like or dislike about mathematics as a subject?

HA27: What I like about mathematics is the availability of different methods of solving a single problem. What I dislike about mathematics is that what might seem mathematically correct to you might be proved wrong or improved further by somebody.

Interviewer: What is solving a mathematical problem?

HA27: Analysing the problem situation and determining a solution to the problem.

Grade 11 students' problem-solving strategies for non-routine mathematical problems

The strategies used by grade 11 students to solve the six non-routine mathematical problems were: Unsystematic Guess, Check and Revise (GCR(unsys)); Systematic Guess, Check and Revise (GCR(sys)); Systematic Listing (SL); Looking for patterns (LP); Consider a simple case (SC); Modelling (MD); Logical reasoning (LG); No logical reasoning (NLG); Trial-and-Error (TE) and Use a Formula (F). Some students could apply more than one strategy to solve a problem. Table 5 illustrates the frequency of strategy used by the systematic, utilitarian and exploratory believers.

Table 5. Students' Problem-Solving Strategies According to Belief

Strategy	Systematic		Utilitarian		Exploratory		Neutral		Total
	f	%	F	%	f	%	F	%	
GCR(unsys)	133	41.7	421	38.8	998	41.2	35	48.6	1587
GCR(sys)	22	6.9	76	7.0	168	6.9	1	1.4	267
SL	13	4.1	53	4.9	130	5.4	2	2.8	198
LP	0	0.0	0	0.0	3	0.1	0	0.0	3
SC	25	7.8	99	9.1	191	7.9	4	5.6	319
MD	29	9.1	110	10.1	155	6.4	4	5.6	298
LG	22	6.9	63	5.8	144	5.9	2	2.8	231
NLG	73	22.9	242	22.3	570	23.5	24	33.3	909
TE	1	0.3	10	0.9	40	1.7	0	0.0	51
F	1	0.3	10	0.9	22	0.9	0	0.0	33
Total	319	100.0	1084	100.0	2421	100.0	72	100.0	3896

Systematic believers' problem-solving strategies

GCR(unsys) (41.7%) and NLG (22.9%) had the highest frequency of use while GCR(sys) (6.9%) and LG (6.9%) had the lowest frequency of use. Figures 1 and 2 show how a typical systematic believer approached the two randomly selected problems.

2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon?

(Adopted from Greenes et al., 1986, p. 12)
HA 37

ANSWER	ROUGH CALCULATIONS
$= \frac{16 \text{ km}}{10 \text{ km more}} \text{ a gallon}$ $(10+16) \quad 26 \text{ km} = x$ $\frac{16x}{16} = \frac{26}{16}$ $x = 1,625 \text{ gallons}$ <p>a gallon = R 2,00</p> $1,625 \text{ gallons} = x$ <p>new car now 26 per gallon</p> $\text{a gallon} = 26 \text{ km}$ $x = 1500 \text{ km}$ $x = \frac{15000}{26}$ $x = 576,92 \text{ gallon}$ $\text{a gallon} \times 576,92$ $= R 1153,85$ $= R 1875 - R 1153,85$ $= R 721,15 \text{ (money saved)}$	$16 \text{ a gallon} = 16 \text{ km}$ $x = 15000 \text{ km}$ $x = \frac{15000}{16}$ $x = 937,5 \text{ gallons}$ <p>a gallon = cost R 2,00 gasoline</p> $937,5 = x$ $937,5 \times 2 = x$ $R 1875 \text{ gasoline}$

Figure 1. Systematic believer's approach to question 2.

The student applied logical reasoning and presented the solution process in a series of logical steps. The student's approach to problem-solving was effective, as they considered the whole picture of the problem situation.

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting up the fish they caught:

- Thabo caught more than Joel.
- Annah and Refilwe together caught as many as Joel and Thabo
- Annah and Thabo together did not catch as many as Refilwe and Joel.

Who caught the most? Who came in second, third and fourth?

(Adapted from Callejo & Vila, 2009, p. 115)

H#37

ANSWER	ROUGH CALCULATIONS
<p style="margin-left: 20px;"> MD GCR (unsys) 2 </p> <p>Refilwe caught the most Thabo came in second Joel came in third Annah came in fourth</p>	<p style="margin-left: 20px;"> $T > J$ $A + R = T + J$ $T + A < R + J$ less </p>

Figure 2. Systematic believer's approach to question 6.

The student modelled the situation using inequalities and then presented the names of people in order of the number of fish they caught. The student did not explain how they deduced the order of the people. Deductively, when a student was faced with a blockage and could not use the models of their creation, they resorted to the “unsystematic guess, check and revise” approach to problem-solving.

Utilitarian believers' problem-solving strategies

GCR(unsys) (38.8%) and NLG (22.3%) had the highest frequency of use while SC (9.1%) and GCR(sys) (7.0%) had the lowest frequency of use. Figures 3 and 4 show the strategies applied by a typical utilitarian believer to the two randomly selected problems.

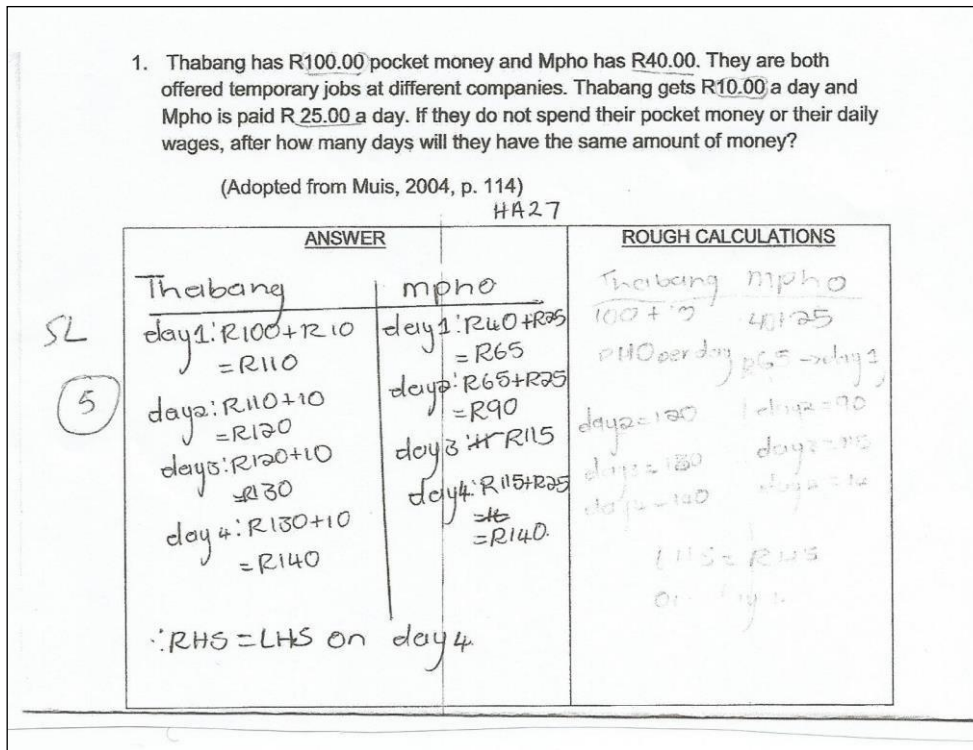


Figure 3. Utilitarian believer’s approach to question 1.

The student systematically calculated the amounts of money each person had from day one to day four. This empirical search for the solution was effective in this particular case because the number of days taken to have the same amount of money was small. Otherwise, the approach would have proved inconveniently long with more days involved.

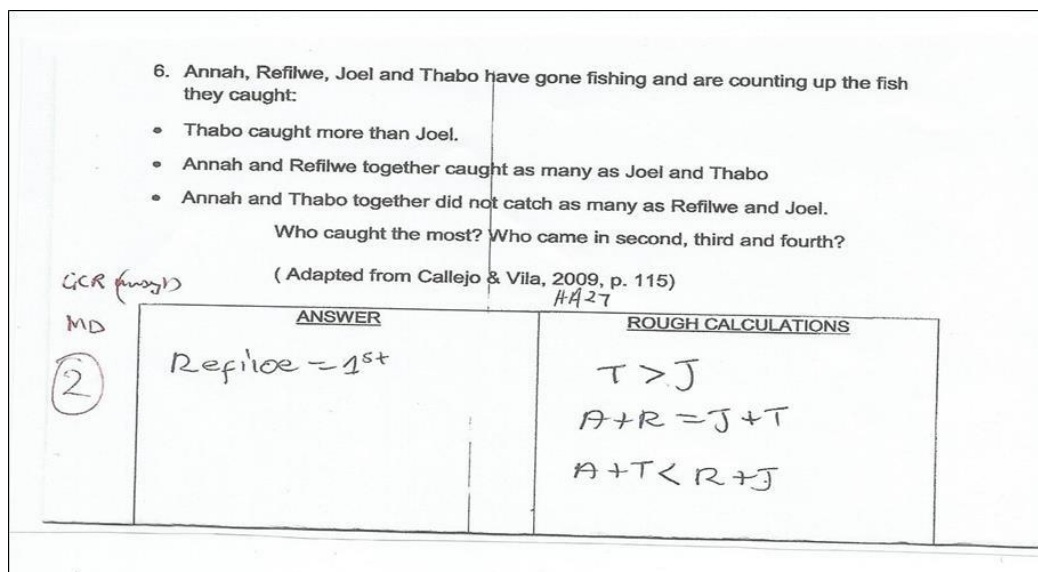


Figure 4. Utilitarian believer’s approach to question 6.

Like Figure 2, the student modelled the situation using inequalities and thereafter abandoned their models in favour of guesswork. This problem-solving behaviour might be due to limiting the amount of time and effort one engages in problem-solving or lacking confidence and trust in one’s ability and

models to solve the problem. This problem-solving behaviour could be driven, possibly, by the belief that the answer is more important than the process. Similarly, Kolovou et al., (2008) discovered that through guesswork, students could at first give a correct solution which they will later change to an incorrect solution.

Exploratory believers' problem-solving strategies

GCR(unsys) (41.2%) and NLG (23.5%) had the highest frequency of use while GCR(sys) (6.9%) and MD (6.4%) had the lowest frequency of use. Figures 5 and 6 show how a typical exploratory believer approached the two randomly selected problems.

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money?

(Adopted from Muis, 2004, p. 114)

ANSWER		ROUGH CALCULATIONS	
<p>SL (4)</p> <p>Day four (4)</p>	HDS	<p>Day 1 Mpho $40 + 25 = 65$ Thabang $100 + 10 = 110$</p> <p>Day 2 Mpho $65 + 25 = 90$ Thabang $110 + 10 = 120$</p> <p>Day 3 Mpho $90 + 25 = 115$ Thabang $120 + 10 = 130$</p> <p>Day 4 Mpho $115 + 25 = 140$ Thabang $130 + 10 = 140$</p>	

Figure 5. Exploratory believer's approach to question 1.

Similar to Figure 3, the student systematically searched for the solution from day one to day four.

3. There are 18 animals in Thabo's farmyard. Some are chickens and some are cows. Thabo counted 50 legs in all. How many of the animals are chickens and how many are cows?

(Adopted from Muis, 2004, p. 114)

ANSWER		ROUGH CALCULATIONS	
<p>GCR(unsys) MD (4)</p> <p>we have 7 cows and 28 legs of the cow.</p> <p>we have 11 chickens and 22 legs of the chickens.</p>	HDS	<p>Cows </p> <p>Chickens " " " " " " " " " " " " " " " "</p>	

Figure 6. Exploratory believer's approach to question 3.

The student modelled the situation using diagrams and then presented a solution to the problem. The student did not show how the solution was obtained. This problem-solving behaviour could be likened to the systematic and utilitarian believers' approaches to Figures 2 and 4, respectively.

NMPS mean scores of systematic, exploratory and utilitarian believers

On average, the students scored approximately 12 out of 30 marks, with a standard deviation of 0.72 on the non-routine mathematics problem-solving test. This revealed a poor performance on NMPS. The systematic believers had the highest NMPS mean score of 10.18 out of 30 with a standard deviation of 4.08. The utilitarian believers had an NMPS mean score of 10.01 out of 30 with a standard deviation of 4.00. The exploratory believers had the lowest NMPS mean score of 9.86 out of 30 with a standard deviation of 3.87. The one-way analysis of variance revealed that the differences in NMPS mean scores among systematic, exploratory and utilitarian believers were not statistically significant at 0.05 level of significance [$F(2; 611) = 1.132$; $p = 0.323 > 0.05$] (see Table 6. More robust tests of equality of means, the Welch [$F(2; 126.948) = 1.094$; $p = 0.338 > 0.05$] and Brown-Forsythe [$F(2; 183.509) = 1.079$; $p = 0.342 > 0.05$] ascertain that the differences in problem-solving mean scores among the systematic, exploratory and utilitarian believers were not statistically significant.

Table 6. ANOVA Result of Differences in Problem-Solving Mean Scores

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	34.826	2	17.413	1.132	.323
Within Groups	9400.902	611	15.386		
Total	9435.728	613			

Relationship between belief systems and problem-solving strategies

There was a weak statistical linear relationship between the students' MRBS and their strategies for NMPS ($r = 0.346$). As such, the students' MRBS could explain an 11.97% change in approach to NMPS, and vice versa ($R^2 = 0.1197$). An analysis of the relationship between each category of belief systems and strategy used to problem-solving yielded similar results. Specifically, there was a weak statistical linear relationship between the students' systematic MRBS and their approach to NMPS ($r = 0.361$). As such, the students' systematic MRBS could only explain a 13.03% change in approach to NMPS, and vice versa ($R^2 = 0.1303$). There was also a weak statistical linear relationship between the students' exploratory MRBS and their approach to NMPS ($r = 0.203$). The students' exploratory MRBS explained a very small percentage change in their approach to NMPS ($R^2 = 0.0412$) and vice versa. The relationship between the students' utilitarian MRBS and their approach to NMPS was also weak ($r = 0.313$). As such, the students' utilitarian MRBS could explain a 9.80% change in their approach to NMPS, and vice versa ($R^2 = 0.0980$).

Discussion

The students' predominant mathematics-related belief systems

The students held the mathematics-related beliefs understudy as explained by their high beliefs mean scores. The high beliefs mean scores could be attributed to students choosing from the closed-form questionnaire, socially acceptable beliefs that do not reflect the actual beliefs they hold (see Reddy et al., 2006). To counteract this weakness, the researcher inferred some students' mathematics-related

beliefs from their written responses to the test and their responses to the open-ended questionnaire and interviews. The phenomenon of a student holding multiple belief systems explains the interwovenness and complexity belief systems. A similar finding was expounded on by Daskalogianni and Simpson (2001) and Di Martino (2004). The possibility of a single belief belonging to different belief systems in different students exacerbates the complexity of students' behaviour on problem-solving. Though the predominant belief system could, to a higher extent, guide student behaviour on problem-solving, we could not undermine the influence of other underlying salient belief systems to student problem-solving behaviour. Hence, from every problem-solving behaviour exhibited by a student, we inferred the possible beliefs behind it.

The students' problem-solving strategies for non-routine mathematical problems

Systematic believers' predominant strategies for problem-solving were GCR(unsys) and LG. When resolving real-life problems that could not be solved by readily available procedures, systematic believers, often applied logical reasoning and presented the solution process in a series of logical steps. When resolving problems that do not fit into any usual exercises done before and faced with a blockage, systematic believers resorted to "unsystematic guess, check and revise" approach to non-routine mathematical problem-solving. For the most part, when the systematic believers applied guesswork, they presented non-sensible solutions. Probable cause was the failure to monitor and control their solution process.

Utilitarian believers' predominant strategies for problem-solving were GCR(sys), SC and MD. They often approached problem-solving by repeating the given information. Possible reasons for this problem-solving behaviour could be attempting to fit the problem to the usual, routine problems solved previously and looking for algorithmic procedures that could be applied to solve the problem. Utilitarian believers also searched for the solutions systematically and modelled problem situations using diagrams and inequalities. Whenever they failed to use the models created, they often resorted to guessing work. They engaged less in logical or rational reasoning to verify their solutions. As a result, they occasionally presented non-logical solutions.

Exploratory believers' predominant strategies for problem-solving were NLG, SL and TE. They frequently approached problem-solving by modelling the problem situations, looking for patterns and creating systematic lists. Sometimes they considered the "whole picture" of the problem situation and resultantly created correct mathematical models. On solving problems, they often made a "rushed" response to resolving the problems. Consequently, they made oversight of some conditions stated in the problem situation and created incomplete or incorrect mathematical models. This behaviour could be attributed to limiting the time one takes to resolve a problem. Like systematic and utilitarian believers, exploratory believers often abandoned their models and resorted to "unsystematic guess, check and revise" and "non-logical reasoning" approaches to problem-solving in the face of blockage. This finding resonates with that of Kolovou et al. (2008). Owing to this problem-solving behaviour, exploratory believers, at times, could not look back to check whether the conditions stated in the problem were met, the solution obtained makes sense or if the problem has been solved. Similar findings on students' failure to monitor and control their solution processes were reported by Callejo and Vila (2009).

Some problem-solving behaviour was common among students holding different predominant belief systems. For example, all students resorted to "unsystematic guess, check and revise" and "non-logical reasoning" in cases of blockage on resolving non-routine mathematical problems. Systematic and utilitarian believers habitually presented non-logical solutions due to little engagement in metacognitive monitoring and control. "Trial-and-error" and "use a formula" strategies were commonly applied by utilitarian and exploratory believers. Some behavioural traits were unique to a specific belief system. For example, exploratory students frequently applied different approach strategies to resolve a single mathematical problem. They could switch from one approach strategy to the other as they explore

the problem situation. They considered the “whole picture” of the problem situation, and the pattern of thinking to be applied was determined by their belief system.

Although the systematic believers’ problem-solving mean score was slightly higher than the mean scores of the exploratory and utilitarian believers, the differences among the mean scores were not statistically significant. Low mean scores among the groups (systematic, exploratory and utilitarian believers) could be attributed to poor student performance on solving non-routine mathematical problems and limited application of a variety of problem-solving strategies. This finding accords with the findings of DBE (2019) and Reddy et al. (2006) in South Africa, Mogari and Lupahla (2013) in Namibia, and Kolovou et al. (2008) in the Netherlands.

Relationship between belief systems and problem-solving strategies

Students’ systematic MRBS explained more percentage change in students’ approach to NMPS compared to their utilitarian and exploratory MRBS. That explained the relatively smaller percentage changes in approach to NMPS. Overall, the relationship between students’ MRBS and their approach to NMPS revealed a weak positive linear relationship between the two constructs. This might mean that if a student develops more desirable beliefs in mathematics learning and problem-solving, their strategy application to problem-solving will likely improve as well. This finding corroborates the findings of Mason (2003), Schoenfeld (2016) and Spangler (1992) but is not in agreement with the findings of Callejo and Vila (2009) and Goldin et al. (2009), who described the relationship between the two constructs as complex.

Conclusion

The 11th graders had different predominant belief systems (systematic, exploratory, and utilitarian) that could explain their behaviour in problem-solving. They apply a variety of problem-solving strategies, namely Unsystematic Guess, Check and Revise; Systematic Guess, Check and Revise; Systematic Listing; Looking for patterns; Consider a simple case; Modelling; Logical reasoning; No logical reasoning; Trial-and-Error and Use a Formula. Although the differences in problem-solving mean scores among the systematic, exploratory and utilitarian believers were not statistically significant, the systematic belief system explained more percentage change in the students’ strategy for problem-solving compared to utilitarian and exploratory belief systems. A positive relationship between belief systems and strategies for problem-solving could mean that if students improve on developing positive mathematics-related belief systems, their approach to problem-solving (in terms of appropriate selection and effective application of strategies) could relatively improve as well, and vice versa. A weak relationship between the two constructs could mean that a relatively significant improvement in health mathematics-related belief systems could yield a very small improvement on an effective approach to non-routine mathematics problem-solving.

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Appendices

Appendix A: Extract of the mathematics beliefs questionnaire

	Strongly agree	Agree	Uncertain	Disagree	Strongly disagree
1. I am not good in maths	1	2	3	4	5
2. Those who are good in maths can solve any problem in a few minutes	1	2	3	4	5
3. I know I can do well in math	5	4	3	2	1
4. I think I could handle more difficult math	5	4	3	2	1

Appendix B: Marking rubric for the non-routine mathematics problem-solving test

Description of the problem-solving strategy	Code
The student makes use of all the relevant information in the problem situation to solve the problem; The student connects the given information and presents correct mathematical expressions or statements; The student applies correct and suitable problem-solving strategies; The solution is relevant to the problem; The student verifies the solution correctly; The student explains or justifies the solution satisfactorily.	5
The student can translate the problem situation into mathematical expressions or statements but fails to use the statements to solve the problem; The student uses strategies efficiently and can explain partially correct the solution; The student is only partially able to make connections between/among concepts; The student presents a solution that is partially related to the problem; The student verifies the solution partially correct.	4
The student understands the problem situation partially; The student starts to solve the problem appropriately but later diverts to applying incorrect procedures; The student fails to use the strategies and concepts correctly and efficiently; The student presents a solution that contains errors and misconceptions; The student fails to give a clear explanation or argument; The student uses a wrong procedure to verify the solution.	3
The student can write down the important information in the problem situation but fails to use it to solve the problem; The student represents the problem situation using wrong expressions and statements; The student applies a wrong procedure to solve the problem; The student presents a solution that is full of errors, mistakes and misconceptions; The student fails to explain or justify the arguments correctly; The student applies an incorrect procedure to verify the solution.	2
The student represents the problem situation using wrong expressions and statements; The student applies a wrong procedure without understanding the underlying concepts; The student presents an answer only without showing the procedures used; The student's solution is not related to the problem; The student fails to explain the solution; The student fails to verify the solution.	1
The student leaves the answer sheet blank; The student fails to understand the problem; The student copies parts of or the whole problem without attempting to solve it; The student fails to write down any appropriate information on the problem.	0

Adapted from Mabilangan et al. (2011) and Khashan (2014).

Appendix C: Non-routine mathematics problem-solving test

1. Thabang has R100.00 pocket money and Mpho has R40.00. They are both offered temporary jobs at different companies. Thabang gets R10.00 a day and Mpho is paid R 25.00 a day. If they do not spend their pocket money or their daily wages, after how many days will they have the same amount of money? (Adapted from Muis, 2004, p. 114)
2. My old car goes 16 km on a gallon of gasoline. I drive about 15 000 km a year. If gasoline costs R 2.00 per gallon, how much money can I save if I buy a new car that gets 10 km more to the gallon? (Adapted from Greenes et al., 1986, p. 12)
3. There are 18 animals in Thabo's farmyard. Some are chickens and some are cows. Thabo counted 50 legs in all. How many of the animals are chickens and how many are cows? (Adapted from Muis, 2004, p. 114)
4. Some people had afternoon tea in a cafe which only sold tea and cakes. The tea cost R3.00 a cup, and the cakes cost R 5.00 each. Everyone had the same number of cups and the same number of pieces of cakes. The bill came to R133.00. Can you find out how many cups of tea each person had? (Adapted from Burton, 1984, p. 80)
5. There are some rabbits and some rabbit hutches. If seven rabbits are put in each rabbit hutch, one rabbit is left over. If nine rabbits are put in each rabbit hutch, one hutch is left empty.

Can you find how many rabbit hutches and how many rabbits there are? (Adapted from Burton, 1984, p. 64)

6. Annah, Refilwe, Joel and Thabo have gone fishing and are counting the fish they caught:

- Thabo caught more than Joel.
- Annah and Refilwe together caught as many as Joel and Thabo
- Annah and Thabo together did not catch as many as Refilwe and Joel.

Who caught the most? Who came in second, third and fourth? (Adapted from Callejo & Vila, 2009, p. 115)