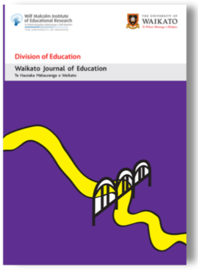

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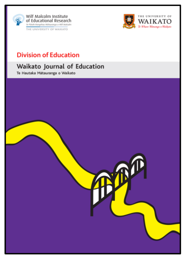

Use of theories and models in geometry education research: A critical review

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Abstract

The aim of this article is to provide a critical review of the theories and the model used in the field of geometry education research. The article critically discusses van Hiele's theory, Fischbein's theory of figural concepts, Duval's theory of figural apprehension, the Spatial Operational Capacity (SOC) model by Wessels and van Niekerk, and the Sfard's commognition theory. The van Hiele's theory proposed a sequential order of development through which the learners construct their understanding of geometry concepts. Fischbein's theory of figural concepts suggested that a geometric figure is always comprised of a visible representation and a concept. Duval's theory of figural apprehension underscored the heuristic value of a geometry figure for solving geometry problems. The SOC model by Wessels and van Niekerk emphasised the importance of instructional design incorporating a variety of physical and mental objects to work with to develop geometry concepts. Finally, the article discusses Sfard's commognition theory that emphasises the communicative function of language in developing geometry concepts. There are two major concerns highlighted with respect to these theories and the model. Firstly, these theories and model emphasise the development of the two-dimensional geometry concepts, neglecting the development of the concepts of three-dimensional geometry. Secondly, these theories and the model fail to acknowledge the multilingual context of geometry class. The article aims to highlight the dearth of studies that explore the multilingual context of geometry class and calls for future studies in this direction.

Keywords

Mathematics, geometry education, geometry concepts, van Hiele theory, figural concepts, figural apprehension, instruction, multilingual context

Introduction

Mathematics has been argued as one of the most powerful subjects that influences an individual's ways of dealing with various spheres (private, social or civil) of his/her life (Anthony & Walshaw, 2007). Thus, teaching and learning of mathematics aims to equip the learner with the knowledge and skills required to deal with the mathematical demands of everyday life. Developing logical and systematic thinking along with flexibility, criticality and creativity is the core of mathematics education (Ministry

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of Education, 2007). Within mathematics, geometry is one of the two pillars (Atiyah, 2002), algebra as the other pillar. Developing a sound understanding of geometry concepts is vital to succeed in mathematics (Education Review Office, 2018). The focus of geometry education is to develop concepts and skills that enable the learners to make sense of the world around them (Jones & Mooney, 2003).

According to the New Zealand Curriculum (Ministry of Education, 2007), Measurement and Geometry is a strand in the learning area of Mathematics and Statistics. Within this strand, there are four sub-strands: measurement, shapes, position and orientation, and transformation. The teaching and learning of geometry involves "recognizing and using the properties and symmetries of shapes and describing position and movement" (p.26). According to the Programme for International Student Assessment (PISA) report (2012), the achievement levels of students in geometry were found to be very low in New Zealand (Ministry of Education, 2015). To improve geometry concepts, the cultural knowledge of learners provides plentiful opportunities (Anthony & Walshaw, 2007). These reservoirs of learners' cultural knowledge are often accessible only when the learners are provided with a supportive environment that appreciates their socio-cultural identities and provides space for their language(s). Therefore, acknowledging the diverse context of New Zealand is of utmost importance for teaching and learning of geometry.

New Zealand is a nation of *superdiversity*[1] in terms of ethnicities and the languages of people (Spoonley & Bedford, 2012). With the increased rate of immigration in New Zealand (New Zealand Immigration, 2018) from various parts of the world, the presence of multiple languages in every domain of social life is foreseeable (Statistics New Zealand, 2013). Acknowledging and appreciating the diverse language backgrounds of students has been recognised to promote overall positive learners' identities and thus, inspire them to actively engage with their learning process (European Commission, 2015; Lo Bianco, Slaughter, & Schapper, 2016). In such a linguistically super diverse nation, it is arguably crucial to ponder on the processes through which learners construe their understanding of geometry concepts while negotiating meanings of their constructions in a milieu of multiple languages.

In light of the diverse context of New Zealand, the aim of the present paper is to look at the research literature critically on the use of theories and models in geometry education research. The next section discusses the Van Hiele (1959) theory, the Fischbein's theory of figural concepts (1993), the Duval's (1995) theory of figural apprehension, the Spatial Operational Capacity model by Wessels and Van Niekerk (2000), and the Sfard's (2008) commognition theory. The discussion highlights the focus on the development of two-dimensional geometry concepts, neglecting the three-dimensional geometry concepts. The critical review also draws attention to the dearth of literature concerning the exploration of the multilingual context in the field of geometry education research. The article concludes by presenting an argument that the multilingual context presents plentiful possibilities for conducting future researches.

Literature review

This section presents a critical review of the theories and the model that has been used for developing an understanding of how learners construct their meanings of geometry concepts. The literature reveals that studies situated in the school education context majorly refer to three theories and a model. These theories and model include the van Hiele's (1959/1985) theory, the theory of figural concepts by Fischbein (1993), the Duval's (1995) theory of figural apprehension, the Spatial Operational Capacity model by Wessels and van Niekerk (2000), and Sfard's commognition theory (2008). The intent of this section is to provide a brief discussion of these theories with their limitations.

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[1] Vertovec (2007) used the term super-diversity to account for the complex diversities in terms of ethnicities arising because of immigration in Britain. Spoonley and Bedford (2012) used the same concept as superdiversity (without hyphen) to explain the New Zealand context.

Van Hiele's theory (1959/1985)

This theory is based on a constructivist approach, largely on the lines of the Piagetian theory of cognitive development. Pierre van Hiele and his wife Dina van Hiele developed the sequential theory for explaining how learners develop their geometry concepts. The progress of the learners at each level is dependent upon their prior experiences, knowledge and mastery at the previous level. The learners progress through five sequential thought levels in their developmental trajectories, given their prior appropriate instructional experience. Fuys, Geddes, and Tischler (1988) translated the van Hiele theory and the levels in English, which have been validated by van Hiele (Van Hiele, 1999). These thought levels are stated in Table 1.

Table 1. Thought levels in Van Hiele's Theory

Level 0: The student identifies names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance. Level 1: The student analyses figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding, measuring, and using a grid or diagram). Level 2: The student logically interrelates previously discovered properties/ rules by giving or following informal arguments. Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems. Level 4: The student establishes theorems in different postulational systems and analyses/compares these systems.

Note: Adapted from "The van Hiele model of thinking in geometry among adolescents," by Fuys, D., Geddes, D., & Tischler, R., 1988, *Journal for Research in Mathematics Education*, Monograph, 3, p. 5. Copyright 1988 by the National Council of Teachers of Mathematics.

According to van Hiele, for each geometric level, there are five phases of learning. These phases of learning help the learner to move from one thought level to the next. The first phase of learning is the *information/inquiry* phase. In this phase, the learners elicit information about the objects of study in terms of their observations and questions. The teacher and learner engage in a conversation to mark the beginning of the next phase. The second phase of learning is *guided orientation*. This phase of learning involves the use of carefully sequenced activities by the teacher to guide the learners' orientation or attention to notice specific features of the geometric structures like similarity and symmetry. The *explicitation* marks the third phase of learning. This phase is concerned with the refined use of vocabulary learned in previous phases to express the inherent, implicit structures of objects. Until phase three, teacher's instructions play an important role in providing direction to learners, prompting them to notice important features of the structures, and helping them to articulate those observations in technical/geometry vocabulary. In the fourth phase of learning, the teacher's role becomes covert in providing the help needed. This is the phase of *free orientation*. At this stage, students encounter multiple yet different tasks to realise the internal structure of objects to resolve these tasks without the teacher's explicit facilitation. Yet the abilities to synthesise different properties to make a unified whole, and to make use of various methods to resolve problems are not achieved by this stage. These abilities are achieved in the last phase of learning that is the *integration* phase (Fuys et al., 1988).

Van Hiele made use of didactics to focus on the teaching of the structures of geometric objects. According to him, the progression from one thought level to another is more reliant upon the instructions that the learner receives rather than the learner's maturity or age. Hence, providing appropriate instructions according to the sequenced phases of learning and thought level is of utmost importance. There exists a vast literature that supports van Hiele's theory for its instructional focus (e.g., Sinclair & Moss, 2011; Stumbles, 2018).

In addition to learning phases, van Hiele also emphasised the role of *insight* in developing geometry concepts. A person is said to have an insight if "the person (a) is able to perform in a possibly unfamiliar situation; (b) performs competently (correctly and adequately) the acts required by the situation; and (c) performs intentionally (deliberately and consciously) a method that resolves the situation" (Hoffer, 1983, p. 205). In other words, having insight enables the learners to know about their actions in terms of what to do, why and when. The van Hiele instructional approach has been integrated with technologies within the dynamic geometry environments such as Logo and GeoGebra to promote the development of geometry concepts (e.g., Clements, Battista, & Sarama, 2001; Korenova, 2017; Venturini & Sinclair, 2017).

Yet, the theory is not free from drawbacks and criticisms. The theory has been, firstly, criticized for emphasising that the development takes place in a sequential manner. Research shows that the same student may possess different van Hiele levels for different geometry concepts simultaneously (Battista, 2009; Bleeker, Stols, & Putten, 2013). With specific concern for the development of shapes and their representation, Pyshkalo, Russian psychologist and educator, drew heavily on Van Hiele's theory to develop an instructional plan for primary school learners. Pyshkalo (1968) found that "familiarizing second graders with solids enabled them to reach the second level (van Hiele level 1), surpassing the progress of seventh graders in the traditional schools" (as cited in Hoffer, 1983, pp. 209-210). A recent body of research provides evidence in accordance with Pyshkalo (Bruce & Hawes, 2015; Gagnier, Atit, Ormand, & Shipley, 2017; Sinclair & Bruce, 2014) and therefore questions the sequential order of thought levels.

The second criticism of the van Hiele theory is related to the first criticism. Van Hiele developed the theory to help students to construct geometry concepts by providing appropriate instructions as per the thought levels. However, Ness and Farenga (2007) have criticised the theory by arguing that it is very difficult to identify the van Hiele level for a learner. As it has been stated earlier, a learner may be at different van Hiele levels for different geometry concepts.

Thirdly, the studies based on the van Hiele theory attempt to locate the misconceptions about geometry concepts at different stages in a diagnostic manner. Therefore, the focus of these studies is to suggest the development of remedial material (e.g., Gunčaga, Tkacik, & Žilková, 2017). Such an approach does not take account of *how* the concept is developed, rather it is concerned with *what* concept has been developed.

Fourthly, the van Hiele theory focuses on the development of concepts of the Euclidean geometry. It does not account for any developmental trajectory for non-Euclidean geometries (Guven & Baki, 2010). The translation of van Hiele task modules and level descriptors in the Brooklyn College project (Fuys et al., 1988) clearly mentions rectangular figures, their properties, axioms and other aspects as worthy of understanding as part of school geometry. This overemphasis with planar (two-dimensional) shapes and related concepts has resulted in confusion regarding *diagrams* and *representation* with geometry concepts of shapes (Battista, 2009). Irrelevant characteristics of diagrams are often attributed to the geometry concept (Clements & Battista, 1992). For example, the narrowness of a triangle is often attributed as a reason for not considering it as a triangle (Devichi & Munier, 2013).

Finally, the van Hiele theory undertakes a limited approach to the role of language in the development of geometry concepts. The role of language is restricted in terms of definitions of the geometry concepts of sides and angles (Van Hiele, 1999). The theory situates the role of language within the issue of disharmony in communicating the features and properties of the geometric structure.

Disharmony arises because of misconceptions about the mathematical terms and their meanings. This limited understanding of language is concerned with the use of geometry vocabulary, neglecting the communicational function of language that fosters meaning constructions of geometry concepts.

Fischbein's theory (1993) of figural concepts

Fischbein (1993) proposed that geometric figures are not solely concepts but they have an intrinsic figural nature. That is, a geometric figure is a figure and a concept simultaneously. A figure is a spatial sensory representation that is subjected to figural laws (of closure, proximity, boundedness) whereas a concept is an abstract idea defined by a set of axiomatic conditions. The interplay of these two types of properties of geometric figures results in a conceptual understanding of geometry figures or concepts. The geometric figure is conceptualised in a symbiotic relationship of these two aspects, in which "it is the figural facet which is the source of invention, while the conceptual side guarantees the rigour and the logical consistency of the operations" (Fischbein & Nachlieli, 1998, p. 1195). The interaction between the figure and the concept in conceiving the geometrical concept is influenced by the whole perception of the figure (Gestalt, whole). A difficulty in conceptualising a geometric figure may arise if the figural properties are not in accordance with the conceptual properties of the figure. And this tension may give rise to prototypical figural concepts (Fujita, 2012; Hershkowitz, 1990). That is, learners may not recognise a rectangular quadrilateral as a parallelogram even though they have knowledge of conceptual properties of a parallelogram (Fujita & Jones, 2007; Walcott, Mohr, & Kastberg, 2009).

The theory considers the conceptual development of geometry concepts as merely cognitive concepts with no mention of the role of language in conceptual development. In addition, the studies based on Fischbein's theory of figural concepts have majorly focused on properties of two-dimensional shapes, providing no valuable insights for the conceptual development of three-dimensional shapes (e.g., Fujita & Jones, 2006; Vodušek & Lipovec, 2014; Walcott et al., 2009).

Duval's theory of figural apprehension (1995)

According to Duval (2017), a given representation can be recognised in several distinctive ways depending on the set of rules applied for visual representations. This suggests that to view figures geometrically, a set of rules are always present that must be followed to view the given figure in the geometric sense (as a geometric shape). As a result, a considerable amount of cognitive leap is required to view the figures geometrically as the representations, against their automatic perceptual recognition. Therefore, to perceive a figure geometrically, a learner needs to perceive its figural units in different dimensions. That is, to recognise a geometric shape as a cube, figural units of the cube (3D), its faces (2D), sides (lines, 1D), and the vertices (0D) must be grasped. He argues that it is because of the figural units of the higher dimensions that the perceptual recognition of the figural units of the lower dimensions are blocked. This breaking up of a figure according to different figural units is the process of dimensional deconstruction of shapes. He argues that to learn geometry, one needs to deconstruct dimensionally all 2D shapes and use the figures as heuristics to understand the representations. For understanding a figure in a geometric sense, it must act as a heuristic to evoke cognitive apprehensions (Duval, 1995).

There are four kinds of cognitive apprehensions: perceptual, sequential, discursive, and operative. The *perceptual* apprehension is concerned with the unconscious integration of the figural organisation laws and the pictorial cues that result in a particular visual representation. The *sequential* apprehension is related to the way the representations are deconstructed in terms of its figural units. The process of deconstruction of figural units is based on geometry concepts and properties. The *discursive* apprehension informs us about the details of the figure that cannot be determined without additional information through speech (written and/or oral). Discursive apprehension works in situations when the details of the representations are not clear from the figure. For example, a figure may look like a rectangle but the details about its angles, length of sides and feature of parallelism will conclude if it is one or not. The last apprehension is the *operative* apprehension. Operative apprehensions involve operating with the figure in various ways - dividing it into parts to locate shapes, changing the orientation of the figure, spatially putting it in other places or in other ways, and/or obtaining an insight to a solution of the problem.

Duval suggested that for a figure to function as a geometric heuristic for solving problems, it must evoke a perceptual apprehension and one of the other three cognitive apprehensions. Along with this, Duval argued that to recognise any shape, a learner must also be able to distinguish the physical object (for example, a cardboard template) from its semiotic representation (geometric figure - rectangle). He emphasised the use of sign systems in developing the concept of 'figures' and the underlying operations that work at different levels. It is this triadic structure among the object (3D), mathematical object (what the figure represents, e.g., the shape as a rectangle), and the figure itself (the drawing) that develop the understanding of a shape. Hallowell, Okamoto, Romo, and La Joy (2015) found that learners found it difficult to relate a plane rectangle and plane triangle figure with the solid cylinder and solid cone. Thus, it is not intuitive for young learners to work out these complex relationships among these three aspects to deconstruct the dimensionality. There are underlying mechanisms through which the meaning of a particular figure is substantiated.

Interestingly, the studies situated in Duval's theory of figural apprehension have focused on the construction of concepts of 2D shapes, in spite of acknowledging that the objects used are 3D in nature (e.g., Gómez-Chacón & Kuzniak, 2015; Tanguay & Venant, 2016; Vendeira & Coutat, 2017). Additionally, these studies fail to provide an account on the role of language in mediating the meanings of shapes and their representation, while being situated in the multilingual context (e.g., Arıcı & Aslan-Tutak, 2015; Duroisin & Demeuse, 2015).

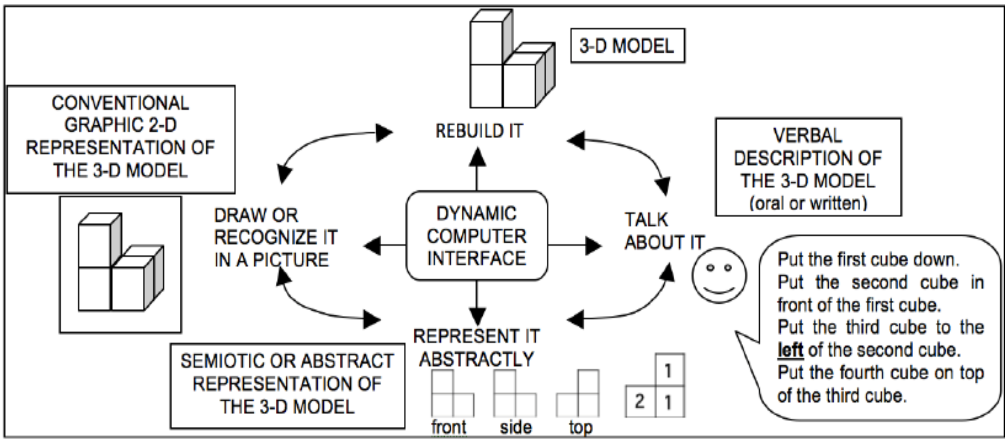
Spatial Operational Capacity Model (2000)

This model was developed by Wessels and van Niekerk (2000) and it combines the works of Yakimanskaya (1991) and van Niekerk (1997) (as mentioned in Sack & van Niekerk, 2009). The spatial operational capacity (SOC) model is based on the assumption that learners should be provided with ample opportunities, that require them to explore and work with a variety of physical and mental objects and their transformation, so as to develop the diverse set of skills required to solve different kinds of geometry problems.

According to the SOC model, there are four categories of variables that contribute to the complexity of a visual image. These are *perception, dimensionality, transformation, and mobility*. The variables of perception include a stimulus that can be present as full-scale images, virtual/real images, conventional graphic images or iconic images. The dimensionality variables include the aspects of the objects that a learner perceives, processes or acts on as part of the whole stimulus that is presented via visual information. The third category of variables includes transformations of shapes. These variables focus on the cognitive processes that are at work while processing the stimulus visually, while it is transformed either positionally, structurally or in both ways. The mobility variable is the fourth variable and is concerned with the mobility of the visual stimulus and whether the stimulus is static, semi-dynamic or dynamic (Sack & Vazquez, 2016).

Based on the SOC model, Sack and Vazquez (2016) conducted a study for a period of seven years in elementary school. There were 14 fourth graders and 11 third graders. The purpose of the study was to explore the development of 2D and 3D geometry concepts. During this study, Geocadabara Construction Box dynamic computer interface instructional design was integrated with the other three sets of models (that are full-scale models, conventional graphic models, and semiotic models) to address the complex nature of teaching and learning in geometry (see Figure 1). The full-scale models (or scaled-down models) are real objects that can be manipulated by the student. The conventional-graphic models represent two-dimensional graphic (2D) representations of the real, three-dimensional (3D) objects. The semiotic models are abstract, symbolic representations that bear no resemblance to the actual objects, for example, floor plan diagrams (Sack & van Niekerk, 2009). They argued for developing an understanding of geometry concepts, a learner needs to develop competencies in these three different representational modes. The instructional plan moves from 3D objects to 2D representations of 3D objects and then to 2D representations of two-dimensional objects (an abstract concept).

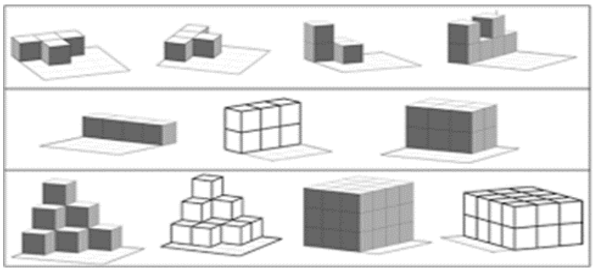
Figure 1. Adopted SOC model



Adapted from "A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children," by J. Sack and I. Vazquez, 2016, p. 7. Copyright 2016 by the J. Sack and I. Vasquez.

There were three major findings. Firstly, the learners were not able to visualise the hidden cubes in 3D structures (made up of unit cubes) as shown in Figure 2.

Figure 2. Pre-interview figures



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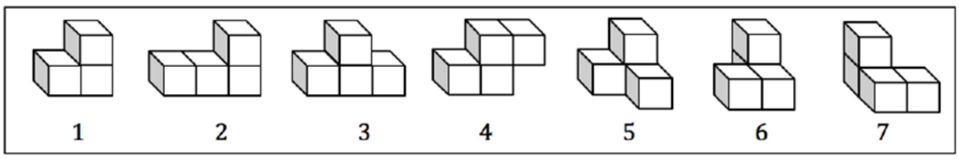
Secondly, learners encountered difficulties in expressing the similarities and differences in 3D structures made up of unit cubes (see Figure 3). The learners experienced ambiguities in communicating and perceiving meanings of terms like horizontal, vertical, etc. (Sack & Vazquez, 2016).

Figure 3. The seven figures from unit cubes



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Thirdly, they found that perspectives, orientations and positions play a crucial role in determining the representations that the learners may build for the 3D models. Accordingly, these aspects influence the conceptual development of geometry concepts of 2D and 3D shapes.

The SOC model underscores the role of language in verbally describing the figures. Yet, the processes through which learners may have sailed through the ocean of different cultural meanings during conversations associated with shapes, have not been considered, in spite of having a highly diverse population. An exploration into the processes through which learners communicated their understanding of geometry concepts, while interacting in diverse linguistic settings, was not conducted. It may have provided valuable insights on how learners navigate through multiple languages to develop their understanding of geometry concepts. The dearth of exploration of multilingual context in the process of development of geometry concepts in visual-spatial abilities, is evident.

Sfard’s commognition theory (2008)

With specific attention to the role of language in the development of geometry concepts, Sfard’s (2008) commognition theory has been widely used to study the communicative role of language (Kaur, 2015; Ng & Sinclair, 2015; Wang, 2016). According to Sfard, cognitive and interpersonal communicational processes are different manifestations of the same phenomenon. Thus, to understand the cognitive processes that enable learners to develop geometry concepts, the communicative processes need to be studied. (Sfard & Kieran, 2001). She argued that for communication to be effective, there are two conditions, firstly it fulfils its communicative purpose by fulfilling expectations based on intentions and secondly the act of communication should have no evidence of a breach. Thus, the effectiveness of the communication depends on the harmony between the speaker’s intentions entailed with expectations on the one hand, and the expected outcome from the listener, on the other. She argued that an act of communication is effective as long as there is no evidence of breach or incongruency between intentions and expectations (Sfard & Kieran, 2001). Sfard’s theory analyses communicational aspects of language in terms of discourses that are present in a mathematics class. Discourse-based view of language attempts to locate the way language operates within social and cultural contexts.

Based on Sfard’s commognition theory, Kaur (2015) explored the communicational aspect of language that mediates the development of geometry concept of triangles with grade 2-3 students. The participants were 7 to 8-year-olds. She found that that the discourse regarding the identification of triangles moved along the proposed order of discourse, firstly the discourse of visual object recognition, secondly the discourse of informal properties, and thirdly the discourse of definitions. Different types of routines and words are used at each level of discourse. Kaur explained the communicational features of language that mediate the development of geometry concepts, yet the communicational space of multilingual context has not been explored. Also, the preoccupation with two-dimensional geometry concepts is evident in the studies (e.g., Presmeg, 2016; Wang, 2016) using Sfard’s commognition theory.

Conclusion

This article identified major theories and models that reflect the research trend in the field of geometry education. The van Hiele theory proposed a sequential developmental order through which the learners develop their understanding of geometry concepts. The second theory discussed in this article is the Fischbein’s theory of figural concepts. This theory suggests that a geometric figure is always comprised of a visible representation and a concept. The interplay of these two aspects defines any geometry figure. After the Fischbein’s theory, the article discusses the theory of figural apprehension by Duval. Duval underscored the heuristic value of geometry figure for solving geometry problems. He argued that a figure functions as a heuristic by eliciting four kinds of cognitive apprehensions. These are perceptual, sequential, discursive, and operative apprehensions. He argued that for a figure to perform as a heuristic, the figure must evoke perceptually and one of the other three figural apprehensions. Followed by the

Duval's theory, the article reviewed the SOC model for the development of geometry concepts. The model emphasised the importance of instructional plan in incorporating a variety of physical and mental objects in developing geometry concepts. Finally, the article discusses the Sfard's commognition theory that emphasises the communicational role of language in developing mathematical concepts.

In all the theories and the model, the preoccupation with the conceptual development of two-dimensional geometry and its related concepts, is evident. Moreover, the critical review of the literature reveals that there is a dearth of studies exploring the multilingual context of geometry classes. Sinclair et al. (2016) published a review article focusing on the research contribution since 2008 in the field of geometry education. Interestingly, the article fails to mention the complexities arising due to the multilingual context of teaching and learning of geometry. The theories fail to acknowledge the processes through which learners negotiate their understanding of the geometry concepts while interacting with others.

It is crucial now, to take a step further in the field of language(s) and mathematics to explore the dynamic nature of language(s) use that promotes the sense-making process of mathematical concepts in a multilingual context. Exploring the multilingual context of geometry class may provide a valuable focus of inquiry. In contemporary times, developing a better understanding of how learners navigate through the milieu of multiple languages to build their understanding of geometry concepts, is of utmost importance. This gap in the literature opens up possibilities for future research. Exploring the conversational processes that enable learners to express their understanding of different geometry concepts, while negotiating their constructions, may help to gain valuable insights for geometry education research.

References

Anthony, G., & Walshaw, M. (2007). *Effective pedagogy in Mathematics/Pāngarau: Best Evidence Synthesis Iteration (BES)*. Wellington, New Zealand: Ministry of Education.

Arıcı, S., & Aslan-Tutak, F. (2015). The effect of origami-based instruction on spatial visualization, geometry achievement, and geometric reasoning. *International Journal of Science and Mathematics Education, 13*(1), 179-200. doi:10.1007/s10763-013-9487-8

Atiyah, M. (2002). Mathematics in the 20th century. *Bulletin of the London Mathematical Society, 34*(01), 1-15. doi:10.1112/s0024609301008566

Battista, M. (2009). Highlights of research on learning school geometry. In T. Craine & R. Rubenstein (Eds.), *Understanding geometry for a changing world* (pp. 91-108). Reston, VA: The National Council of Teachers of Mathematics

Bleeker, C., Stols, G., & Putten, S. V. (2013). The relationship between teachers' instructional practices and their learners' level of geometrical thinking. *Perspectives in Education, 31*(3), 66-78.

Bruce, C. D., & Hawes, Z. (2015). The role of 2D and 3D mental rotation in mathematics for young children: What is it? Why does it matter? And what can we do about It? *ZDM, 47*(3), 331-343. doi:10.1007/s11858-014-0637-4

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In G. D.A. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York, NY: Macmillan.

Clements, D. H., Battista, M. T., & Sarama, J. (2001). *Logo and geometry* (Vol. 10). Reston, VA: National council of Teachers of Mathematics, Inc.

Devichi, C., & Munier, V. (2013). About the concept of angle in elementary school: Misconceptions and teaching sequences. *The Journal of Mathematical Behavior, 32*(1), 1-19. doi:10.1016/j.jmathb.2012.10.001

Duroisin, N., & Demeuse, M. (2015). What role for developmental theories in mathematics study programmes in French-speaking Belgium? An analysis of the geometry curriculum's aspects, framed by van Hiele's model. *Cogent Education, 2*(1). doi:10.1080/2331186X.2015.1049846

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142-157). Heidelberg, Germany: Springer.

Duval, R. (2017). *Understanding the mathematical way of thinking – The registers of semiotic representations* (T. M. M. Campos Ed.). Cham, Switzerland: Springer International Publishing.

Education Review Office. (2018). *Keeping children engaged and achieving in mathematics*. New Zealand: Education Review Office Retrieved from http://www.ero.govt.nz/assets/Uploads/ERO-17763-Teaching-Strategies-that-work-Mathematics-v7.pdf.

European Commission. (2015). *Language teaching and learning in multilingual classrooms*. Retrieved from http://ec.europa.eu/dgs/education_culture/repository/languages/library/policy/policy-brief_en.pdf.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139-162. doi:10.1007/BF01273689

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education, 20*(10), 1193-1211. doi:10.1080/0950069980201003

Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *The Journal of Mathematical Behavior, 31*(1), 60-72. doi:10.1016/j.jmathb.2011.08.003

Fujita, T., & Jones, K. (2006). *Primary trainee teachers' understanding of basic geometrical figures in Scotland*. Paper presented at the PME, Prague. Retrieved from https://eprints.soton.ac.uk/41247/1/Fujita_Jones_PME30_2006.pdf

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education, 9*(1), 3-20. doi:10.1080/14794800008520167

Fuys, D., Geddes, D., & Tischler, R. (1988). *The Van Hiele model of thinking in geometry among adolescents* (Vol. 3). New York, NY: The National Council of Teachers of Mathematics, Inc.

Gagnier, K. M., Atit, K., Ormand, C. J., & Shipley, T. F. (2017). Comprehending 3D diagrams: Sketching to support spatial reasoning. *Topics in Cognitive Science, 9*(4), 883-901. doi:10.1111/tops.12233

Gómez-Chacón, I. M., & Kuzniak, A. (2015). Spaces for geometric work: Figural, instrumental, and discursive geneses of reasoning in a technological environment. *International Journal of Science and Mathematics Education, 13*(1), 201-226. doi:10.1007/s10763-013-9462-4

Gunčaga, J., Tkacik, Š., & Žilková, K. (2017). Understanding of selected geometric concepts by pupils of pre-primary and primary level education. *European Journal of Contemporary Education, 6*(3), 497-515. doi:10.13187/ejced.2017.3.497

Guven, B., & Baki, A. (2010). Characterizing student mathematics teachers' levels of understanding in spherical geometry. *International Journal of Mathematical Education in Science and Technology, 41*(8), 991-1013. doi:10.1080/0020739x.2010.500692

Hallowell, D. A., Okamoto, Y., Romo, L. F., & La Joy, J. R. (2015). First-graders' spatial-mathematical reasoning about plane and solid shapes and their representations. *ZDM: The International Journal on Mathematics Education, 47*(3), 363-375. doi:10.1007/s11858-015-0664-9

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In J. Kilpatrick & P. Nesher (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 70-95). New York, NY: Cambridge University Press.

Hoffer, A. (1983). Van Hiele-based research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 205-227). Orlando, FL: Academic Press.

Jones, K., & Mooney, C. (2003). Making space for geometry in primary mathematics. In I. Thompson (Ed.), *Enhancing primary mathematics teaching* (pp. 3-15). Philadelphia, PA: Open University Press.

Kaur, H. (2015). Two aspects of young children's thinking about different types of dynamic triangles: Prototypicality and inclusion. *ZDM, 47*(3), 407-420. doi:10.1007/s11858-014-0658-z

Korenova, L. (2017). GeoGebra in teaching of primary school mathematics. *The International Journal for Technology in Mathematics Education, 24*(3), 155-160. doi:10.1564/tme_v24.3.08

Lo Bianco, J., Slaughter, Y., & Schapper, A. (2016). UNICEF EAPRO – Suggestions for UNICEF EAPRO strategy (2016-2020) on multilingual education and social cohesion. Retrieved from https://www.researchgate.net/publication/306741873_UNICEF_EAPRO_-_Suggestions_for_UNICEF_EAPRO_Strategy_2016-2020_on_Multilingual_Education_and_Social_Cohesion/citations

Ministry of Education. (2007). *The New Zealand Curriculum*. Wellington, New Zealand: Learning Media Limited

Ministry of Education. (2015). *NMSSA Report 4: Mathematics and Statistics 2013*. Retrieved from New Zealand: http://nmssa.otago.ac.nz/reports/NMSSA-Maths-and-Statistics-2013.pdf

Ness, D., & Farenga, S. (2007). *Knowledge under construction: The importance of play in developing children's spatial and geometric thinking*. Lanham, MD: Rowman & Littlefield Publishers.

New Zealand Immigration. (2018). *Statistics Arrivals*. Retrieved from New Zealand https://www.immigration.govt.nz/documents/statistics/statistics-arrivals-by-month

Ng, O.-L., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM, 47*(3), 421-434. doi:10.1007/s11858-014-0660-5

Presmeg, N. (2016). Commognition as a lens for research. *An International Journal, 91*(3), 423-430. doi:10.1007/s10649-015-9676-1

Sack, J., & van Niekerk, R. (2009). Developing the spatial operational capacity of young children using wooden cubes and dynamic simulation software. In T. Craine & R. Rubenstein (Eds.), *Understanding geometry for a changing world: Seventy-first yearbook* (pp. 141-154). Reston, VA: The National Council of Teachers of Mathematics

Sack, J., & Vazquez, I. (2016). *A 3D Visualization Teaching-Learning Trajectory for Elementary Grades Children* (1st ed. 2016.. ed.). Cham, Switzerland: Springer International Publishing.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42-76. doi:10.1207/S15327884MCA0801_04

Sinclair, N., Bartolini Bussi, M. G., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. *ZDM: The International Journal on Mathematics Education, 48*(5), 691-719. doi:10.1007/s11858-016-0796-6

Sinclair, N., & Bruce, C. D. (2014). *Spatial reasoning for young learners*. Paper presented at the Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education. Vancouver, BC: PME. Retrieved from https://s3.amazonaws.com/academia.edu.documents/40348841/Research_Forum_-_Spatial_Reasoning_in_Young_Children_p.25-26.pdf?AWSAccessKeyId=AKIAIWOWYYGZ2Y53UL3A&Expires=1536465082&Signature=HCLqx4U6RI171k2trunxloe42fE%3D&response-content-disposition=inline%3B%20filename%3DTHEY_ARE_GETTING_MARRIED_TOWARDS_A_DYNA.pdf

Sinclair, N., & Moss, J. (2011). The more it changes, the more it becomes the same: The development of the routine of shapes identification in dynamic geometry environment. *International Journal of Educational Research, 51-52*(3). doi:10.1016/j.ijer.2011.12.009

Spoonley, P., & Bedford, R. (2012). *Welcome to Our World? Immigration and the Reshaping of New Zealand*. Auckland, New Zealand: Dunmore Publishing Ltd.

Statistics New Zealand. (2013). *2013 Census QuickStats about national highlights*. Wellington, New Zealand: Statistics New Zealand. .

Stumbles, R. (2018). Dynamic geometry software within the van Hiele teaching framework. *Australian Primary Mathematics Classroom, 23*(1), 25-30.

Tanguay, D., & Venant, F. (2016). The semiotic and conceptual genesis of angle. *ZDM: The International Journal on Mathematics Education, 48*(6), 875-894. doi:10.1007/s11858-016-0789-5

Van Hiele, P. (1959/1985). The child's thought and geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele* (pp. 243-252). New York, NY: The National Council of Teachers of Mathematics, Inc.

Van Hiele, P. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics, 5*(6), 310-316.

Vendeira, C., & Coutat, S. (2017). *Shape recognition in early school: How to develop the dimensional deconstruction*. Paper presented at the Tenth Congress of the European Society for Research in Mathematics Education, Dublin. Retrieved from https://keynote.conference-services.net/resources/444/5118/pdf/CERME10_0364.pdf

Venturini, M., & Sinclair, N. (2017). Designing assessment tasks in a dynamic geometry environment. In A. Leung & A. Baccaglini-Frank (Eds.), *Digital technologies in designing mathematics education tasks: Potential and pitfalls* (Vol. 8, pp. 77-98). Cham, Switzerland: Springer International Publishing.

Vertovec, S. (2007). Super-diversity and its implications. *Ethnic and Racial Studies, 30*(6), 1024-1054. doi:10.1080/01419870701599465

Vodušek, H. B., & Lipovec, A. (2014). The square as a figural concept. *Bolema: Boletim de Educação Matemática, 28*(48), 430-448. doi:10.1590/1980-4415v28n48a21

Walcott, C., Mohr, D., & Kastberg, S. E. (2009). Making sense of shape: An analysis of children's written responses. *The Journal of Mathematical Behavior, 28*(1), 30-40. doi:10.1016/j.jmathb.2009.04.001

Wang, S. (2016). *Discourse perspective of geometric thoughts* (1st ed. 2016.. ed.). Wiesbaden: Springer Spektrum.

Wessels, D. C., & Van Niekerk, H. M. (2000). Semiotic thinking and the development of secondary school spatial knowledge. *Tydskrif vir Christelike Wetenskap, 3*, 1-16.