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Viewing basic math through the lens of history: Undergraduates' reflective learning in a history-augmented mathematics classroom

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Abstract

This study is aimed at determining first-year university students' reflections when Fibonacci tiling, the ancient Chinese fang cheng procedures, and the ancient Indian meru prastara recursions were introduced as historical snippets in an adventure pedagogy for basic mathematics. Seventy-eight first-year students enrolled in a course in basic mathematics at a University in North Central Nigeria provided composite self-reports in an action research paradigm, describing their reflective learning after exposure to the historical snippets. Qualitative data reduction strategies were used to explore the students' reflections and progress in the course. The results of the study revealed that the introduction of the historical snippets aids in concretizing of concepts, spurring of behavioural engagement in learners, adding of aesthetic value to mathematics, sustaining of students attention, computational ease and effective recall of mathematical procedures. The activities of the cultural and historical augmentation were reported by participants as plays that accomplish real mathematical tasks. The outcome of this study has strengthened the belief that the history-as-a-tool style of mathematics instruction stimulates curiosity and sustains interest in students while establishing meaningful relationships between abstract ideas and practical applications in the context of the real world.

Keywords

Historical snippets; fibonacci tiling; fang cheng; meru prastara; reflective learning; mathematics education

Introduction

Mathematics is the chief among disciplines in terms of its rich cultural and historical roots in the practice of ordinary people. This is because mathematics is always adapting to the needs of the society. To get to its present form, the subject have passed through different furnaces of time, refining it into the rudiments of knowledge now communicated coherently as a field of study. Thus, the history of mathematics is a living and indispensable part of mathematics which should never be mistaken for a shadow of the main structure of the subject. Introduction of history of mathematics in the classroom is necessary for promoting the image of mathematics as a vivid discipline which links to reality

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(Furinghetti, 2012), and serves as a lens for viewing the diverse concepts taught in mathematics. Teaching history of mathematics serves students by humanizing the discipline, making the topics more accessible by connecting them to individuals and individuals' experiences (Troutman & McCoy, 2008). Augmenting routine classroom instruction with relevant and related mathematical facts raises the aesthetic value of the subject.

History of mathematics can be used implicitly and explicitly to enrich mathematics education. On the one hand, the history of mathematics can be used as an implicit resource in the design of activities to adapt some standard concepts to the teaching syllabus, to choose context and to prepare problems and auxiliary sources (Vallhonesta, Esteve, Casanova, Puig-Pla, & Roca-Rosell, 2015). On the other hand the history of mathematics can be used in an explicit way to direct and propose research works using historical mathematics to aid students' understanding of mathematical concepts. History of mathematics helps to develop the analytic and synthetic thought process of students. Ideas generated from historical attachments to mathematical concepts can truly create conflictive situations in which students are encouraged to reflect upon the rules that define their actions when dealing with the concepts (Bernardes & Roque, 2015). Blending history into construction delivery in mathematics results in both sensitive and historical thinking mediated by bodies, signs artefacts and cultural meanings (Guillemette, 2015). This approach gives rise to non-mentalist conception of thought. Generally, students' reflections about the nature of mathematics through history evidently results in enhanced mathematical literacy, increased psychological motivation, interdisciplinarity, and linguistic and transverse competencies. The usage of original sources, for instance, has been shown to be effective in the teaching and learning of mathematics (Guillemette, 2015). For teachers in particular, the conceptual history of mathematics allows them to understand why certain concepts are difficult for students to understand (Heeffer, 2006). An integrated approach in which the teacher employs the history of mathematics in explaining certain concepts delivers the subject beyond the textbook presentation of definitions, symbols and formulas.

Considering this focal role of history of mathematics in mathematics education, several studies into its specific and adaptive relevance have yield guaranteed results. Troutman and McCoy (2008) observed improved students' attitude after introducing culturally relevant math history lessons in an Algebra class comprising of diverse racial background. When history of mathematics was used to each volume formula of frustum pyramids, students found the activities interesting as well as instructing (Butuner, 2015). The implications of most research findings from history-augmented studies indicates that the sustainable conception of mathematics incorporates not only mathematics content knowledge (rules and algorithms) but a meaningful mathematics content base (knowing the why and how of mathematics) and a teachers' attitude towards mathematics (Burns, 2010). Butuner (2015) observed that when researchers discuss their reasons for using history of mathematics in the classroom, they mostly state that history of mathematics will reduce students' anxiety about mathematics, increase their motivation and attitudes towards learning, help them learn the subjects and concepts and reveal the multicultural, developing and dynamic structure of mathematics. The line of thinking from these empirical researches allows mathematics educators to reassert theoretical conjectures and give new ideas for future lines of research in history and epistemology of mathematics (Costa, Alves, & Guerra, 2015). The call for replications of studies imbibing the historical approach to mathematics instruction will continue to be an open-ended one.

In view of the potentials and doors of opportunity presented by the introduction of elements of history in mathematics discussions, this present study seek to add more layers of bricks to the existing structure of research in the area. Specifically, unlike many related studies, this present work unveils the augmentation of classroom instruction in three major topics of foundational mathematics at the undergraduate level. The study threads the path of the illumination approach to the use of history of mathematics in the classroom. In this perspective, the current curriculum is not changed but historical contents are included (Butuner, 2015). For the present study, historical snippets introduced include the contributions of Fibonacci of Pisa to the study of sequence, the fang cheng procedures of The *Nine Chapters on the Mathematical Art* in relation to the study of matrices, and the Meru prastara as it relates to the study of combinations. There is also a unique consideration of undergraduates' reflective learning based on the APOS framework. The study also pursues the fresh approach of reflective free writing in understanding the conception of mathematical knowledge in students.

Theoretical framework

Historical approaches to the teaching of mathematics enact active inquiry in students. The idea is that if students are engaged in activities and learning processes similar to the way mathematicians produce knowledge, the students will develop a deep understanding of Mathematics, as well as of the epistemology and more broadly the nature of the subject (Kjeldsen, 2016). This entails helping students develop informed conceptions of nature of mathematics, that is, understandings about mathematics as a knowledge generation and validation enterprise (Abd-El-Khalick, 2013). Such approaches are rooted in the experiential learning theory which provides a base for the reflective learning framework.

Experiential learning theory (ELT) provides a holistic model of the learning process and a multilinear model of adult development, both of which are consistent with what people know about how people learn, grow and develop (Kolb, Boyatzis, & Mainemelis, 2000). The emphasis on experience in ELT is due to the central role experience plays in the learning process. ELT defines learning as the process whereby knowledge is created through the transformation of experience, with knowledge resulting from the combination of grasping and transforming experience (Kolb, 1984 as cited in Kolb et al., 2000). Basically there are four major stages in instructional approaches based on the ELT. Stage one involves planning the next experience through setting of objectives and learning activities. Stage two is where the actual learning takes place with students actively involved in the learning activities and exploration of learning experience. In stage three, provision is made for reflecting on what actually happened in the delivery. This third stage helps develop metacognition in the student. The fourth stage gives room for discussions and interactions among all elements of the instructional process, with the openness to repeat the same cycle to ensure effectiveness (Chesimet, Anditi, & Ngeno, 2016). When these stages are executed in the mathematics classroom, the outcome is an active interaction, in which students process new information in such a way that it makes sense to them in their own frames of reference-inner worlds of memory, experience and response (CORD, 1999).

In a history augmented learning environment, students discover meaningful relationships between abstract ideas and practical applications in the context of the real world. Concepts are internalised through the process of discovering, reinforcing and relating (CORD, 1999). The augmentation provides room for in-depth reflection and correction to the origins of mathematical concepts taught in class. The heart of all learning lies in the way students process their experiences, critically, and reflective learning involves recalling from experiences and reasoning out how these connect to present and future learning situations (Towndrow, Ling, & Venthan, 2008). Thus, excerpts from history, when integrated into mathematics instruction, from a cognitive foundation of building present and future relationships in the development of key mathematical concepts.

The role played by historical snippets in mathematics instruction delivery can best be described through the APOS framework. APOS theory (Dubinsky & McDonald, 2001) proposes that an individual has to have appropriate mental structures to make sense of a mathematical given concept. The mental structures refer to the likely actions, processes, objects and schema required to learn the concept (Maharaj, 2010). The APOS theory begins with actions and moves through processes to encapsulated objects. These objects, according to Tall (1999), are then integrated into schema consisting of actions, processes and objects which-can themselves be encapsulated as objects. APOS theory assumed that mathematical knowledge which is possessed by someone is the result of interaction outcome with other people and the result of his/her mental constructions in comprehending mathematical ideas. Comprehending of mathematical concept begins by manipulating existing mental construction or manipulating physical objects to form action. When action is done repeatedly, and the individual reflects on it, the outcome is interiorised to become a process, which in turn develops into a cognitive object. The collection of action, process, object and other scheme which is connected integrally and organised structurally in the individuals' thinking is called the schema (Syaiful, Kamid, & Marssal, 2014). According to Dubinsky and McDonald, (2001), the theory makes testable predictions that if a particular collection of actions, processes, objects and schemas are constructed in a certain manner by a student, then this individual will likely be successful using certain mathematical concepts and in certain problem situations.

Based on these proven theoretical foundations, this present study meticulously planned instructional activities around certain historical mathematical snippets to augment the learning of Sequences, Matrices and Combinations. Students were led through manipulations of ancient algorithms and guided in reflection upon their understanding of the concepts.

Empirical studies

Many history-as-a-tool studies are mainly concerned with learning the inner issues of mathematics such as mathematical concepts, theories, methods and formulas (Butuner, 2015). These issues are often analysed through historical films, life stories, renowned problems, names of mathematicians, anecdotes records, and life works of great mathematicians. The target of this style of mathematics instruction is to stimulate curiosity and sustain interests in students, while enriching the repertoire of the mathematics teachers by enhancing their pedagogical content knowledge.

Goktepe and Ozdemir (2013) embarked on a study of 21 8-grade students who were guided through history-enriched mathematics lessons. Worksheets containing two algorithms about calculating the square roots of numbers which were used in history were given to each student. The students were then guided to reach solutions and compare their results to what is obtainable from the calculator. Students' opinions about the integration of the history of mathematics to the lessons were collected through interview form. The results of the study indicated that the history-based procedure was captivating and engaging.

Similarly, in a study on the use of history of mathematics as a methodology in teaching and learning of mathematics at the polytechnic and junior college levels in Singapore, Ho (2008) observed that the approach strengthened students' interest and appreciation, belief, confidence and perseverance in mathematics. A batch of 102 students in Singapore polytechnic who enrolled for the Certificate of Engineering Mathematics (CEM) programme took part in the study. The students provided feedback via their logbooks and filled out an attitude questionnaire at the end of the programme. The results of the Wilcoxon-Mann-Whitney test indicate that the historical approach is more effective.

The work of Costa et al. (2015) presented at the seventh edition of the European Summer University specifically made use of *The Nine Chapters on the Mathematical Art* as a basis of analysing a 10 year old Portuguese child's understanding, reproduction and application of Gaussian Elimination. The epic study uses an explanatory case study to understand how a young child would react to a task on the fang cheng method for solving linear systems of equation, a topic usually taught to students from eighteen in Higher Education. The outcome of the study shows that the ten years old boy was able to solve systems of two, three and four equations with two, three and four unknowns, respectively, using the fang cheng method. There were also indications that the boy would be able to replicate it in similar situations and foremost use it in a-didactical situations.

These works and many more in their class have provided empirical support for the efficacy of the history-augmented approach to mathematics teaching and learning across broad levels. Despite the robustness of some of these studies, they seem to each focus on single aspects of mathematics. This present work, however, integrated history of mathematics into three different aspects of mathematics-sequence, matrix, and combinations. The sample of this present study is an intact undergraduate class that was capable of writing self-reports and reflecting on their own cognitive process. The articulation from the works of Goktepe and Ozdemir (2013), Ho (2008) and Costa et al. (2015) threw up future challenges to investigate the metacognitive processes of students when exposed to historical snippets in mathematics instruction. This present study seeks to heed this call by analysing the reflective learning of undergraduates in a Basic Mathematics class within a historical context.

Research Questions

The following research questions guided this study

1. What are the first-year students' new conceptions of sequence after reviewing Fibonacci's work?
2. What are the first-year students' reflections on matrix algebra after exposure to the fang cheng procedures of the *Nine Chapters of the Mathematical Art?*
3. What are the first-year students' apprehensions of combinations and binomial coefficients after practicing the recursion of the Meru prastara?

Methodology

This study employs an action research design to investigate undergraduates' reflective learning in a history-augmented mathematics classroom. The action research is usually undertaken by the teacher in a school to solve particular problems in the classroom (Emaikwu, 2015). In this particular study, the problem of abstract perception of mathematical concepts was tackled by augmenting each stage of content delivery with historical snippets.

The sample for the study comprises of 78 first-year students of the B.Sc.(Ed.) integrated science programme hosted in the Department of Science Education of the University of Agriculture, Makurdi, Nigeria. The students were enrolled for Basic Mathematics II, a core course in the second semester of the 2015/2016 academic session. The instructor informed the participants of the history-augmentation and they all gave their consent to cooperate in the teaching methodology. An outcome of this consent was their willingness to keep an extra notebook—a kind of student logbook—apart from the main notebook for the course. The extra notebook was meant for personal research and to write their feelings, remarks on the nature of the lesson and other reflective prompts from the instructor. The instructor periodically assesses the students' extra notebooks for periodic feedback.

The aspects of the Basic Mathematics II course considered for this study include sequence, matrix and systems of linear equations, and combinations and permutations. One historical snippet was carefully selected for each aspect of the course. The snippets are presented at the onset of each topic solely for the purpose of individual students' research and progressively developed in subsequent classroom interactions. Within the action research paradigm (observe, reflect, plan and act), both the instructor and students move through each topic conventionally, observing and identifying solution sequences that are prone to error or assumed procedurally too tedious. This approach allows room for class discussions and teacher suggestions leading to a plan of action for appropriate use of the selected historical snippet.

The deliberations on sequence was augmented by the works of Leonardo Fibonacci (c. 1170–c 1250), particularly the Fibonacci numbers. The students were first of all vaguely introduced to the general idea of the Fibonacci sequence and asked to write a two-page summary on the mathematician, the discovery of the sequence and present-day applications of the Fibonacci numbers. In another follow up class, the instructor suggested the idea of the Fibonacci tiles formed by arranging squares whose sides are the Fibonacci numbers. The students were then asked to draw Fibonacci tiles comprising of the first 9 Fibonacci numbers on a drawing paper (the size of paper used in Technical Drawings). After another session, having appraised the submitted drawing papers, the instructor demonstrated how to form the Fibonacci spirat on the Fibonacci tiles by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling. The students were asked to replicate same. All students' history-based tasks were performed as take home activities and discussed at the end of content coverage in the one-hour class period per week.

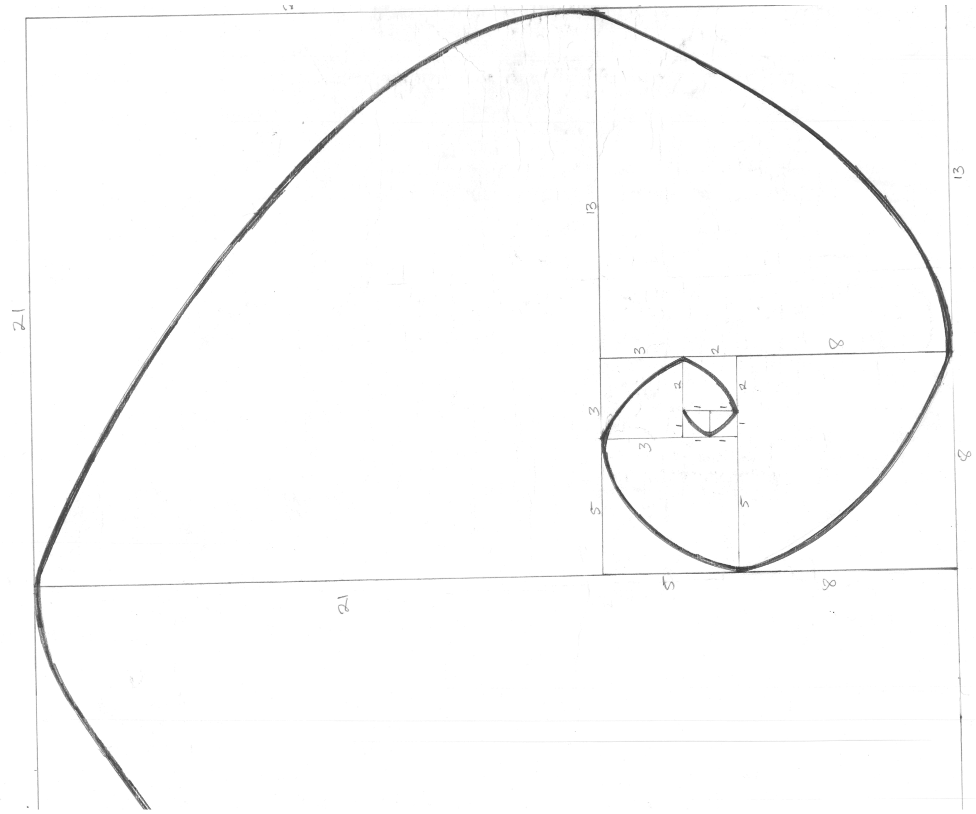

Figure 1. Portion of Fibonacci spriat drawn on Fibonacci tiles comprising of the first nine Fibonacci numbers by a first-year student

The fang cheng (Rectangular arrays) is the 8th chapter of the ancient Chinese mathematics book, *The Nine Chapters on the Mathematical Art* (*Jiu Zhang Suan Shu* in Chinese) composed by several generations of scholars from the tenth to the second century BC (Yuan, 2012). Liu Hui (225–295), one of the greatest mathematicians of ancient China edited and published the book in the year 263. The fang cheng procedures, a set of amazing algorithm for present day Gaussian Elimination, formed the historical snippet for the second topic of the Basic Mathematics II course. The fang cheng was dedicated to solve real-life problems such as calculating yields of grain, numbers of domestic animals, and prices of different products by solving linear equations (Yuan, 2012). Vivid descriptions of the contents of the *fang cheng* are presented in Koh (2012), Grcar (2011a), Yuan (2012) and Grcar (2011b). Liu Hui's detailed commentary about the algorithm for solving the problem was presented by the instructor to the students after solving some systems of linear equations using modern techniques (in several class periods).

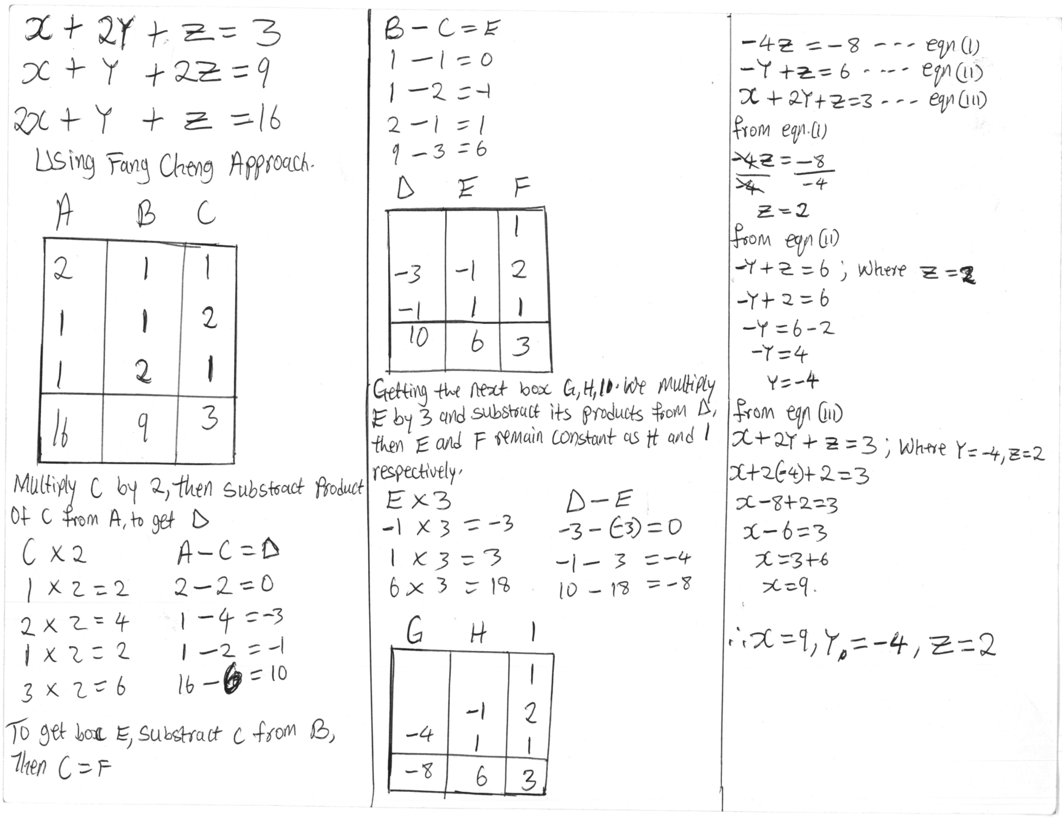

Figure 2. A first-year student's solution to a system of linear equation using the fang cheng procedure

Background emphasis was laid on the fact that in ancient China, this algorithm was implemented by an appropriate number of rods placed within squares on a counting table. Positive coefficients were represented by black rods, negative coefficients were represented by red rods, and the squares corresponding to zero coefficients were left empty. Some open educational resources (OERs) in form of PDF materials were transferred to the students' smartphones and other mobile devices via bluetooth and flashare for in-depth personal studies of the *feng cheng*. Discussions on the *feng cheng* procedures were fully taken in a subsequent class.

The notation of permutations and combinations appeared in early Jaina works in ancient India. Simple combinatorial problems are solved, such as the number of selections that can be made out of a given number of men and women (Sykorova, 2006). Their method of finding the number of combinations is called *Meru prastara* (staircase of mount Meru) and it is a formation of an early Pascal's triangle. The *Meru prastara* rule is based on the following formulae:

$$C_r(n+1) = C_r(n) + C_{r-1}(n)$$

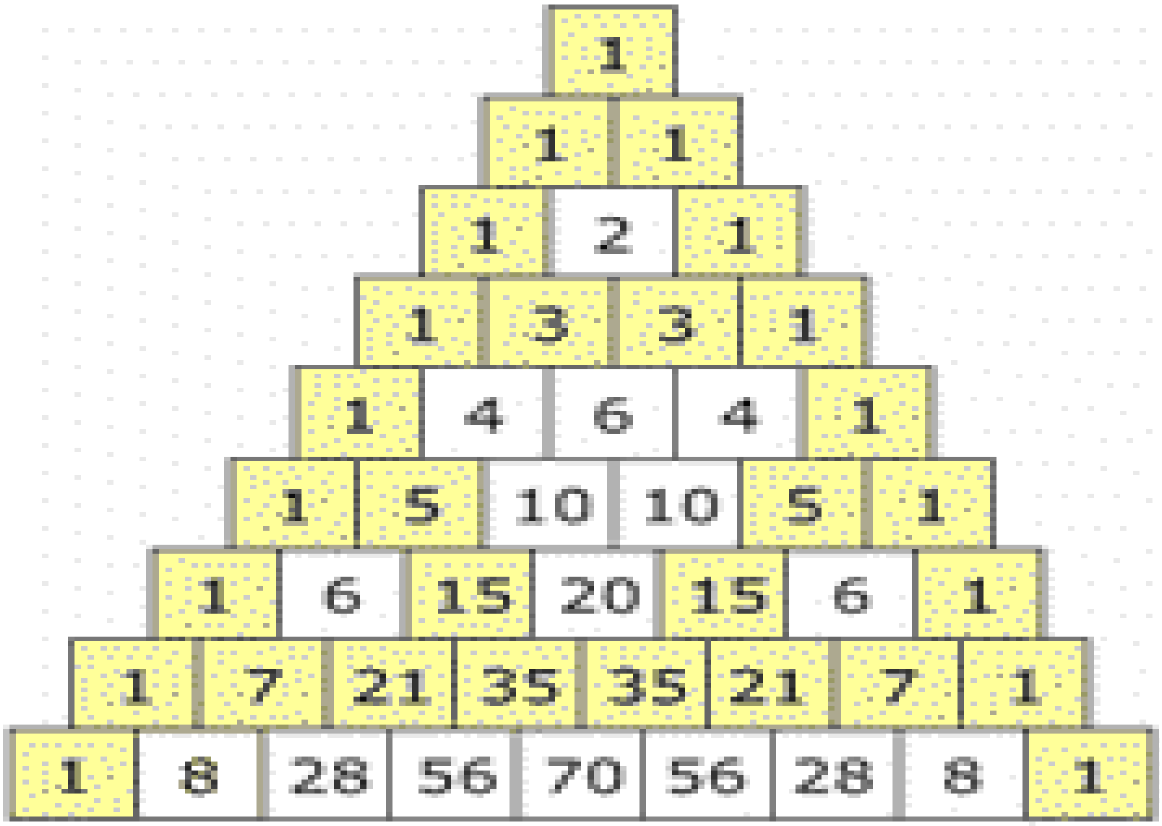

Figure 3. The Meru Prastara

Index	Coefficients					
0	${}^{0}C_{0}$ (=1)					
1	${}^{1}C_{0}$ (=1)	${}^{1}C_{1}$ (=1)				
2	${}^{2}C_{0}$ (=1)	${}^{2}C_{1}$ (=2)	${}^{2}C_{2}$ (=1)			
3	${}^{3}C_{0}$ (=1)	${}^{3}C_{1}$ (=3)	${}^{3}C_{2}$ (=3)	${}^{3}C_{3}$ (=1)		
4	${}^{4}C_{0}$ (=1)	${}^{4}C_{1}$ (=4)	${}^{4}C_{2}$ (=6)	${}^{4}C_{3}$ (=4)	${}^{4}C_{4}$ (=1)	
5	${}^{5}C_{0}$ (=1)	${}^{5}C_{1}$ (=5)	${}^{5}C_{2}$ (=10)	${}^{5}C_{3}$ (=10)	${}^{5}C_{4}$ (=5)	${}^{5}C_{5}$ (=1)

Figure 4. Pascal's Triangle

According to Sykorova (2006), a commentator (Pingla) of the 10th century AD explained the Meru prastara as follows.

i. First draw a square.
ii. Below it, and starting from the middle of the low side, draw two squares. Similarly, draw three squares below these, and so on.
iii. Write the number one (1) in the middle of the top square and inside the first and last squares of each row.

Inside every other square, the number to be written is the sum of the numbers in the two squares above it and overlapping it.

The participants of this study were led through the routine of building the ancient Meru prastara as a historical snippet to aid the learning of Combinations and the Binomial theorem. They were asked to build similar staircase for whatever index needed in any given binomial expansion and report same in their extra notebook.

The instrument for data collection for this study was the students' reflective free-writing report which was basically the accumulated self-reports recorded by the students in their extra notebooks (logbooks). The premise is that engagement in reflective activity can be a driving force in bringing about positive changes in the classroom especially in the areas of teacher-student interactions and in breaking away from the notion, where it exists, that mathematical facts are absolute or unquestionable truths (Towndrow, Ling, & Venthan, 2008). Free writings do not allow time for students to agonize over grammar or spelling; rather, they encourage students to think freely and raise questions about a topic or idea (Urquhart, 2009). In particular, the students' writing allowed for issues to be dealt with by the teacher in the classroom (a core component of action research), which may not have surfaced, at all, through any other means (Towndrow, Ling, & Venthan, 2008). The self-reports helped the students to view the learning of mathematics as an on-going process whereby they stopped at regular intervals to think about what they were doing. The process of data collection is as shown in Figure 5.

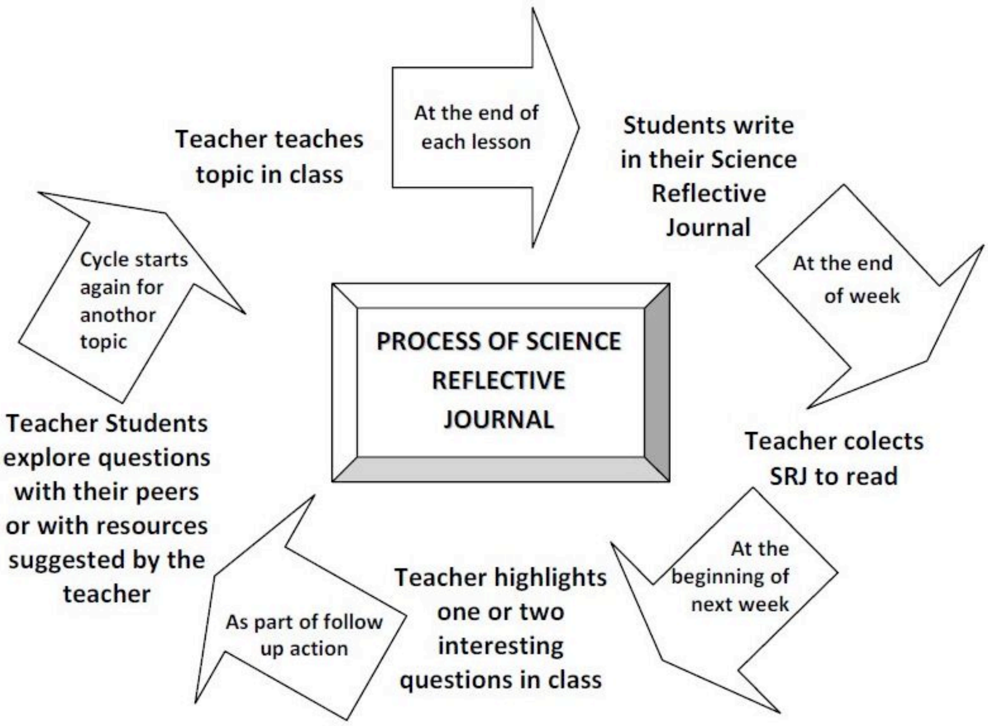

Figure 5. Reflective journal writing (Source: Towndrow et al., 2008)

The tool for data analysis and presentation is simple percentage. The patterns of students' reflections on their own learning were reported with respect to specific prompts by the instructor. The students' self-reports were analysed using qualitative data reduction strategies in order to manage, categorize and interpret data to identify themes (McMohan & Garza, 2017).

Results and Discussions

The results of this study are presented and discussed according to the research questions.

Research Question One

What are first-year students' new conceptions of sequence after reviewing Fibonacci's work?

Three central themes emerged from the reflective reports of first-year students on the Fibonacci tiling activities. These are concretizing of the concept of sequence, spurring of behavioural engagement in the students, and the evidence of easy learning and easy recall.

The qualitative analysis of the participants' reports revealed that 62 percent of first-year students conceived the Fibonacci tiling activities as concretising their learning of the concept of sequence of numbers. This outcome affirms that abstract mathematical relationships are reflected or instantiated, in various forms and at different levels in the concrete structure of the physical world (Abah, 2016). One of the first-year students, S11, reported thus: "Fibonacci tiling activities have given me a concrete understanding of the Fibonacci sequence because it deals with a systematic patterning for easy comprehension".

Along with other participants of the study, S11 observed that even in its most advanced form, mathematics is rooted in reality with applications in everyday life.

The reports also revealed that 81 percent of first-year students experienced increased behavioural engagement in the study of sequence of numbers. Behavioural engagement in mathematics refers to students' disposition to manage their own learning by choosing appropriate learning goals, using their existing knowledge and skills in mathematics to direct their learning, and selecting learning strategies appropriate to the task in hand (Organization for Economic Co-operation and Development–OECD, 2004). To do this, they must be able to establish goals, persevere, monitor their progress, adjust their learning strategies as necessary, and overcome difficulties in learning (Iji, Abah, & Anyor, 2017). In this regard, first-year student wrote:

> I have also learnt this Fibonacci sequence is not only applied in mathematics but also in nature, art and music. Based on my personal research, I have come to realise that the sequence aids investors to predict stock prices. (S12)

Emphasising similar essence of driving personalised learning, S8 reflected thus: "Personally, my interaction with the Fibonacci tiles has aided my learning of the Fibonacci sequence and gave me the privilege to know about Leonardo Bonacci".

These expressions agree with Abd Wahid and Shahrill (2014), who assert that behavioural engagement is expressed in dimensions such as attentiveness, diligence, time spent on task and non-assigned time spent on task. S17 added: "The Fibonacci tiling activity gives me a lot of hints and suggestions for finding more number patterns on my own".

The review of Fibonacci's work by the first-year students results in easy learning and easy recall. This pattern was reported by 59 percent of the students. Bearing witness to this experience, participants S13, S9 and S1 respectively have these to say:

> … by knowing how the Spirat moves, it is easy for me to recall that the sequence is gotten as a result of adding a preceding number to the current number in that sequence.

> The Fibonacci tiling has open my eyes to a great deal of large numbers and how sequences proceed infinitely
>
> It is a unique way of learning sequence.

Research Question Two

What are the first-year students' reflections on matrix algebra after exposure to the fang cheng procedures of *The Nine Chapters on the Mathematical Art*?

An examination of the reflective self-reports of first-year students revealed that the exposure to the fang cheng approach to linear algebra adds aesthetic value to the learning of mathematics, sustains learners' interest, and serves as a more simplified method.

History of mathematics places mathematics in real time and location, thereby adding aesthetic value to the subject (Abah, 2016). Eighty one percent of the first year students who participated in this study provided this line of thought in their reflective reports. Pointing to a sense of admiration, participant S8 reported thus: "The ancient Chinese used the fang cheng approach without being aware that it was in fact a great breakthrough in mathematics".

More pointedly, S1 Wrote: "I admire the method of the ancient Chinese because it is simple and easy to apply".

Similarly, first-year student puts it this way:

> They [the ancient Chinese] thought critically and developed the fang cheng approach that even traders and farmers can use. … indeed, the fang cheng approach in solving matrix algebra has helped me tremendously in developing basic mathematical principles and applying them correctly. (S7)

These expositions agree with Lockhart (2002) who opines that the cultural and aesthetic perspectives to mathematics education enable students to pose their own problem, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration and to cobble together their own explanations and proofs. Such was the experience of one of the first-year students who explained that it took time to comprehend the procedures fully, but later reported: "I started understanding it bit by bit until it became clear to me" (S17).

Indeed, linking the teaching of mathematical concepts to the histories behind them sets learners on the path of mathematical discovery (Yevdokimov, 2006).

Still, 93 percent of the students reported that exposure to the fang cheng procedures sustained their interests in matrix algebra. Participants S2, S11 and S16, respectively wrote:

> It is a very interesting way of solving mathematical problems, especially matrix.
>
> The idea of the fang cheng approach towards resolution of matrix algebra has really helped me to appreciate and comprehend the simplicity in complex linear systems.
>
> It helped me to learn linear equations. It also motivated me to learn more on solving other system of linear equations using the fang cheng approach.

With respect to simplification, 88 percent of participants concur in their reflections that the introduction of the fang cheng procedures lightened their burden in tackling matrix algebra. In this regard, for instance, S8 observed:

> The fang cheng has helped my learning of matrix algebra because before now it was quite difficult with numerous steps one has to follow before arriving at the correct answer. It has been a real shortcut with the fang cheng method.

This form of appraisal by students re-echo the earlier findings of Costa et al. (2015) who found that a 10 year-old child was able to manipulate systems of linear equations using this historical approach. Other participants followed suite in their reported reflections:

> It is shorter and it is more convenient. (S12)
>
> It helps me to solve more complex and long chain matrix algebra more accurately and faster. (S3)
>
> I love the method used by the ancient Chinese because it is simple and easy to apply. … The ancient fang cheng method made my knowledge about matrix algebra to grow very wide. (S6)

Research Question Three

What are first-year students' apprehensions of Combinations and Binomial Coefficients after practicing the recursion of the Meru prastara?

The analysis of first-year students' reports indicates outcomes of computational ease, concretising abstract procedures, and easy recall of the recursion for finding the binomial coefficients.

On computational ease occasioned by the introduction of the Meru prastara, Participant S3 was among 75 percent of the class who reported this pattern:

> The meru prastara has brought to my understanding the simplest and quickest way of expanding binomial terms.

Similarly, S5 and S11 respectively reported these:

> By applying the method of the ancient Indians, I can now solve binomial expressions with ease.
>
> Without a shadow of doubt, knowing about the Meru prastara has first given me an impression that the binomial expression can be done for whatever value of *n* without stress.

The analysis of students' reports also indicates that 66 percent of the participants of this study attested to the Meru prastara aiding in the concretising of a hitherto abstract procedure. This implies that evidently, historical connections give students the opportunity to model mathematical problem in its existential context, to carry out analysis of the learning materials and discover mathematical properties that are completely new (Yevdokimov, 2006). In this respect, participant S8 wrote: "Since the meru prastar*a* gives me a clear pictorial view of Pascal triangle, I can draw up the Pascal triangle more orderly and faster than ever before".

First-year student S7 puts it this way:

> The Meru prastara helped me in understanding the binomial expansion easily, because I now understand the fact that for you to get the correct answer, you need to note that first, $n+1$ terms are there in it; secondly, power of the first term is reducing along; thirdly, power of the second is increasing along.

Similarly, S10 reports: "Now counting the number of blocks horizontally gives you the number of terms of the binomial expression and by counting the blocks diagonally we still get the power of the expression in question".

Obviously, first-year students apprehend the formation of the binomial coefficients through the Meru prastara as "a play that accomplishes a real task".

After practicing the recursion of the Meru prastara, first-year students reported that the approach makes it easy to recall the binomial recursion. 74 percent of the participants reported patterns similar to participants S14 and S15, who respectively wrote:

> The Meru prastara has helped me in finding the coefficient of each term in a binomial expansion as compared to the formula method or other approaches.

> With the help of the Meru prastara approach, I don't think I will be able to forget how to form a Pascal triangle.

One dynamic aspect of this study that cannot be ignored is the storytelling context within which the teaching and learning took place. By connecting the topics to the students through the background narrative of the historical snippets, the approach was truly able to engage their emotions and help them understand the power of the ideas being explored (NrichMaths, 2017). The particular story about the Jania civilisation of ancient India stressed the fact that mathematical tales are universal and enhance the globalisation of cultural knowledge (Hourani, 2015). The first-year students even joked about the possibility of a great mathematical discovery one day coming out of the local, cultural practices of their own people, may be not in their lifetime, but in centuries to come. The activities of this study humanize Basic Mathematics by presenting the interdisciplinary connections between mathematics and other worlds of thought (Goral & Gnadinger, 2006; Toor & Mgombelo, 2015). The students were made to see first-hand that mathematics is indeed an embodiment of tradition.

Conclusion

This study has attempted to present an in-depth report of undergraduates' reflective learning when classroom instruction in Basic Mathematics was augmented with cultural and historical artefacts. Specifically, the Fibonacci tiles, the fang cheng procedure, and the Meru prastara, were respectively deployed to strengthen the learning of sequences, matrix algebra, and binomial combinations by first-year students enrolled for the course. Experiential learning theory and the APOS theory were explored as theoretical foundations for the study. Vivid presentation of reports, coded and deduced from the participants' written self-reports indicates that the augmentation assists students in easy recall of concepts and procedures, sustains the learners' interest, adds aesthetic value to the course, and simplifies computational routines.

Based on the findings of this study, it is recommended that mathematics educators in particular and teachers in general consider the epistemological and cultural approach in the design and delivery of instruction. Introducing ideas from a wide cultural background into classroom teaching not only demystify abstract concepts, but foster a sense of admiration of past contributors to knowledge and build openness and global intellectual relationship in learners. Likewise, the behavioural engagement occasioned by history-augmentation draws on the idea of active student participation and includes involvement in academic, social or extracurricular activities which are considered crucial for achieving positive academic outcomes.

The specific usage of the historical snippets deployed in this study hold broad implications for classroom practices in mathematics education across all levels. The fang cheng procedure, for instance, reduce the tedious steps of the conventional co-factor approach to solving systems of linear equations. The background stories and histories surrounding the snippets can also lighten up classroom interaction while augmenting other teaching strategies, offering teachers flexibility in enriching the instructional process. The snippets also provide linkage to the roots and foundational development of concepts which can concretize mathematical learning. Similarly, the visual appeals of the snippets lend them as tools for aiding the teaching of students known to have mathematics learning difficulty.

The students use of logbooks to record their own learning reflections as exemplified in this study can be readily adopted even outside the mathematical subject-matter domain. Such a planned effort can go beyond ordinary academic assessment to providing substantial feedback that can be used to reinforce instructional strategies in any subject area.

In view of the findings of this study, future studies may seek to investigate the full effect of these combined historical snippets on students' academic achievement, specifically in terms of measurable scores and Grade Point Averages.

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